Kinematics of Fluid Flow, Parts I - V

Jim Price

Woods Hole Oceanographic Institution

508-289-2526

http://www.whoi.edu/science/PO/people/jprice/

Kinematics of Fluid Flow I:
Lagrangian and Eulerian Representations.

To Start: Our goal in this opening section is to define what we mean by 'fluid flow', a phrase we will use repeatedly, and to consider how a fluid flow is like and unlike solid particle dynamics. A second and more substantial goal is to erect a coordinate system[1] that will be suitable for analyzing fluid flows quantitatively. The definition of a coordinate system is a matter of choice, and the questions to consider are more in the realm of kinematics than dynamics. Nevertheless, this definition of a coordinate system is a crucial step that is right at the heart of what makes fluid dynamics a challenging subject. In fact, the dynamics of the fluid flows that we will consider can be characterized as straightforward classical physics built upon the familiar conservation laws - conservation of mass, (recti)linear momentum, angular momentum and energy - and a few others. In this regard, the physics of fluid flow that we will consider are no different from the physics of classical, solid particle mechanics. However a fluid particle is in a literal sense pushed and pulled around by its surroundings (other fluid particles or boundaries) in a way that solid particles may not be (but this depends upon the phenomenon, of course). At the same time a fluid particle may be quite dramatically strained and stretched and become so intermingled

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[1]By coordinate system we mean something more fundamental than the ususal notion of Cartesian or curvilinear coordinates, for example.

with its fluid surroundings as to effectively lose its identity. It is this complete (one is tempted to say intimate) interaction between a fluid particle and its surroundings that characterizes fluid flows and that distinguishes fluid dynamics from solid particle dynamics generally. Thus in a full blown fluid mechanics problem, we have to solve for the fluid motion of the entire fluid domain at once; we generally can not understand or predict the motion of one fluid parcel without understanding the fluid motion over the entire domain, effectively. For even the simplest problems this often will be a huge task that can be approached in full only with the aid of a (very big) computer that is carefully programmed to follow the rules of fluid motion. Approximate solutions of fluid flow often yield crucial insight, and many widely used techniques of applied mathematics were developed to make useful approximate solutions of fluid flow. The challenges of classical and geophysical fluid mechanics stem mainly from the very complex three-dimensional and time-dependent kinematics that characterize most fluid flows, rather than from exotic (non-classical) physics. An understanding of fluid kinematics is thus an essential starting point for the study of fluid flows.

Let's suppose that our task is to observe the fluid flow throughout a three-dimensional domain that we will denote by R^3. Some day your domain will be something grand and important - the earth's atmosphere or perhaps an ocean basin - but to start let's choose something modest and accessible, the fluid flow in a teacup. The fundamental questions that we will consider next are the same for all fluid flows, large and small.

By fluid flow we mean the motion of fluid material throughout the entirity of the three-dimensional space defined by the walls and the free surface of our (teacup) domain. That is, the phrase 'fluid flow' is meant to conjure up the mental image of the entire fluid volume in motion, rather than a single piece in isolation.

The Lagrangian (or Material) Coordinate System. If we intend to make measurements we will invariably do so by discrete (vs. continuous) means. One possibility is that we observe the motion of individual fluid bits, or 'parcels', that can be identified by dye concentration or some other marker that does not interfere with their motion. Because fluid motion may vary on what can be very small spatial scales, we will have to consider the motion of correspondingly small parcels. Let's denote the initial time by t_o and the corresponding initial position of a marked parcel by $\boldsymbol{\xi_o}$. We somewhat blithely assume that we can determine the position of that specific parcel at all later times, t, to form the parcel

$$\text{trajectory} \ \ (\text{or pathline}) = \boldsymbol{\xi}(\boldsymbol{\xi}_o, t). \tag{1}$$

A fluid flow can be envisioned to map the points $\boldsymbol{\xi_o}$ into the points $\boldsymbol{\xi}$ at later times. We will assume that this mapping is unique in that adjacent parcels will never be split apart, and neither will one parcel be forced to occupy the same position as another parcel. Thus given a $\boldsymbol{\xi_o}$ and a time, we will presume that there is a unique $\boldsymbol{\xi}$. We must assume that a parcel can be taken to be as small as is necessary to meet these requirements, and that a fluid is, in effect, a continuum, rather than made up of discrete molecules. With these conventional assumptions, the mapping of points from initial to subsequent positions (the trajectory) can be inverted, at least in principle. Thus, given a $\boldsymbol{\xi}$ and a time, we assume that there is a corresponding unique $\boldsymbol{\xi_o}$.

The velocity of a parcel, oftern termed the 'Lagrangian' velocity, is just the time rate change of it's position,

$$\mathbf{V}_L(\boldsymbol{\xi_o},\, \mathrm{t}) = \frac{d\boldsymbol{\xi}}{dt} = \lim_{\delta t \to 0} \frac{\boldsymbol{\xi}(\boldsymbol{\xi_o}, t + \delta t) - \boldsymbol{\xi}(\boldsymbol{\xi_o}, t)}{\delta t}, \tag{2}$$

where the limit process will be discussed shortly. To specify *where* this velocity was observed we will have to carry along the parcel trajectory (1) as well.

Each trajectory must be tagged with it's unique $\boldsymbol{\xi_o}$ and thus for a given trajectory $\boldsymbol{\xi_o}$ is a constant. Though $\boldsymbol{\xi_o}$ is constant for a given parcel, we have to keep in mind that our coordinate system is meant to describe a continuum defined over some domain, and that $\boldsymbol{\xi_o}$ in principal varies continuously over the entire initial domain of the fluid. Thus when we need to consider the domain as a whole, $\boldsymbol{\xi_o}$ has the role of being the independent, spatial coordinate. This kind of coordinate sytem in which parcel position is the fundamental dependent variable is often referred to as a Lagrangian coordinate system, and also and perhaps more aptly as a 'material' coordinate system.

The acceleration of a fluid parcel is just

$$\frac{d\mathbf{V}(\boldsymbol{\xi_o}, t)}{dt} = \frac{d^2\boldsymbol{\xi}}{dt^2} \tag{3}$$

and note that here and above we have written the time rate of change as an ordinary derivative (and we have also dropped the subscript L since it will be redundant in the circumstances to follow). From the fact that we are differentiating $\boldsymbol{\xi}$ it should (and must, really) be understood that we are asking for the time rate change of the position or velocity of a specific parcel, and thus that we are holding $\boldsymbol{\xi_o}$ fixed during this differentiation. Given that we have defined and can compute the acceleration of a fluid parcel, we go on to presume that Newton's laws apply to a fluid parcel in exactly the form used in classical (solid) mechanics, i.e.,

$$\frac{d^2\boldsymbol{\xi}}{dt^2} = \mathbf{F}/\rho, \tag{4}$$

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F is the force per unit volume imposed upon that parcel, and ρ is the mass density of the fluid. Note that this is *exactly* the form we would use to write the momentum balance for a solid particle. Thus if we observe that a fluid parcel undergoes an acceleration, then we can infer that there had to have been an applied force on that parcel. It is on this kind of diagnostic problem that the Lagrangian coordinate system is most useful, generally. On the other hand, if our task was to compute the trajectory of fluid parcels - the forward problem - then we have to specify the force, **F**, acting on each parcel. This can be particularly difficult - often intractable - in a Lagrangian system except in special cases (but more on this below).

Before going much further it may be helpful to consider a very simple but concrete example of a flow represented in the Lagrangian framework. Let's assume that we have been given the trajectories of all the parcels in a one-dimensional domain R^1 with spatial coordinate y by way of the explicit formula

$$\xi(\xi_o, t) = \xi_o(1 + 2t)^{1/2} \tag{5}$$

where $t_o = 0$. Once we specify the starting position of a parcel, $\xi_o = \xi(t = 0)$, this handy little formula[2] tells us the y position of that specific parcel at any later time. It is most unusual to have so much information presented in such a convenient way, and in fact, this particular 'flow' has been concocted to have just enough complexity to be interesting for our purpose here, but is without physical significance. The velocity of a parcel (or we could say 'speed' since we are in one space dimension) is then

$$V(\xi_o, t) = \xi_o(1 + 2t)^{-1/2} \tag{6}$$

and the acceleration (and also the force per unit volume) is just

$$\frac{d^2\xi}{dt^2} = -\xi_o(1 + 2t)^{-3/2}. \tag{7}$$

Given the initial positions of four parcels, let's say $\xi_o = [0.1 \quad 0.3 \quad 0.5 \quad 0.7]$ we can readily compute the trajectories and velocities from (5) and (6) (Figure 1). Note that the velocity depends upon the parcel initial position, ξ_o. If V did not depend upon ξ_o, then the flow would necessarily be spatially uniform, i.e., all the fluid parcels in the domain would have exactly the same velocity. The flow shown here has the following

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[2]The notation used here could be confusing. When we separate a list of variables by commas as $\xi(\xi_o, t)$ on the left hand side, we mean to emphasize that ξ is a function of ξ_o and t. When variables are separated by operators, as $\xi_o(1 + 2t)$ on the right hand side, we mean that the variable ξ_o is to be multiplied by the sum $(1 + 2t)$.
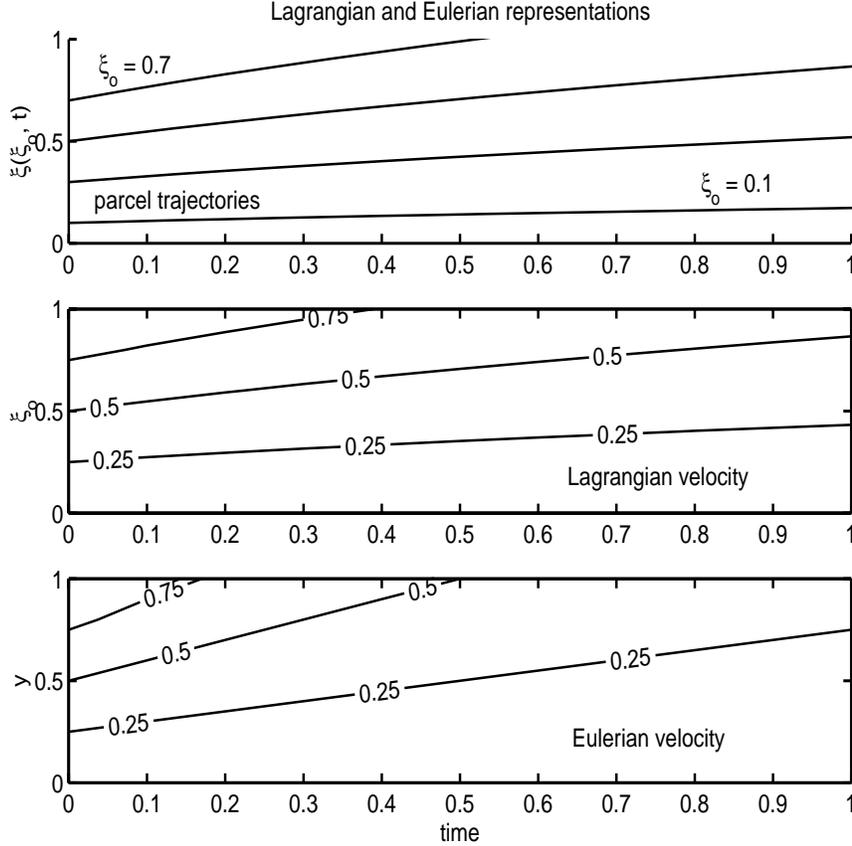
Figure 1: Lagrangian and Eulerian representations of the one-dimensional, time-dependent flow defined by (5). (Upper) The solid lines are the trajectories $\xi(\xi_o, t)$ of four parcels whose initial positions were $\xi_o = 0.1$, 0.3, 0.5, 0.7. (Middle) The Lagrangian velocity as a function of time and initial position (the lines here are contours of constant velocity). (Lower) The Eulerian velocity field computed by solving for $V(y, t)$ (and again the lines are of constant velocity).

form: all parcels shown (and we could say all of the fluid in $y > 0$) are moving in the direction of positive y; parcels that are at larger y move faster; all of the parcels in $y > 0$ are decelerating in the sense that their speed decreases with time, and the magnitude of the deceleration increases with y.

If, as presumed in this example, we are able to track parcels at will, then we can sample as much of the domain and any part of the domain that we may care to investigate. In any real experiment this would be a bit problematic; we could not be assured that any specific portion of the domain will be sampled unless we launched a parcel there. Even then, the parcels may spend most of their time in regions we are not particularly interested in sampling, a hazard of Lagrangian measurement.

Consider the information that this Lagrangian representation provides; in the most straightforward way it shows *where* fluid parcels released into a flow at a given time and position will be found at some later time. If our goal was to observe how a fluid flow carried a pollutant, say, from a source (the initial position) into the rest of the domain, then this Lagrangian representation would be ideal. We could simply release (or tag) parcels over and over again at the source position, and then observe where the parcels (and presumably the pollutant) were subsequently carried by the flow. By releasing a cluster of parcels we could observe how the flow distorted or strained the fluid. Similarly, if our goal was to measure the force applied to the fluid, then by tracking parcels and measuring their accelerations we could estimate the force directly via the equation of motion, (4) (it is very hard to envision a force attached to parcels in just the way indicated by (7)). These are important and common uses of the Lagrangian coordinate system but note that they are all related in one way or another to the measurement of fluid flow rather than to the calculation of fluid flow.

If our goal was to carry out a forward calculation, i.e., to compute rather than observe parcel trajectories, then we would have to be able specify the force, F, acting on the fluid and as we hinted before, this can be very awkward in a Lagrangian system. In part the reason is that F on a parcel will very likely depend upon the spatial variations of the pressure and the velocity in the vicinity of the parcel. This information is generally not immediately available in a Lagrangian system since the fluid surrounding a given parcel will probably not have a simple relation to the initial position i.e., we can not simply differentiate with respect to ξ_o, the independent variable. Differentiation with respect to the dependent variable ξ gives what is usually a highly nonlinear and complex relation. We can always construct a map of the fluid velocity in the vicinity of a given position, but then we may as well admit that we are working in an Eulerian system (which we will consider next). Thus, although the Lagrangian equation of motion is simple and familiar, nevertheless its implementation in a multi-dimensional continuum may be extremely difficult. In practice, Lagrangian models are well-suited to forward calculations in only rather special circumstances. In this course we will examine only two Lagrangian models in which the interactions among 'parcels' are particularly simple, the interesting but rather eccentric case of a finite number of interacting free vortices (which simply advect one another about) and an acoustic plane wave (in which the parcel displacements are very small and do not become tangled). While we will not try to solve full Lagrangian systems, it is often highly desirable to compute the trajectories, accelerations, etc. of fluid parcels in order

to diagnose the dynamics of a fluid flow, and this we will consider in the following sections.

The Eulerian (or Field) Coordinate System. If tracking fluid parcels is impractical, say because the fluid is opaque, then we might choose to observe the fluid flow by means of current meters that we could implant at fixed positions, say **x**. The essential component of every current meter is a transducer that converts fluid motion into a readily measurable signal - e.g., the rotary motion of a propeller or the doppler shift of a sound pulse. Regardless of the details, a fluid velocity measurement is intended to give the speed and direction of fluid parcels that move through the (fixed) control volume sampled by the transducer at a given time,

$$\mathbf{V}_E = \mathbf{V}(\mathbf{x}, t) \tag{8}$$

where the bold notation indicates a vector in R^3. The crucial difference between this velocity $\mathbf{V}_E$ and the velocity derivable from parcel tracking, $\mathbf{V}_L$, is that the position in Eqn (8) is the arbitrary (our choice), fixed position of the current meter, where the position in the Lagrangian velocity of Eqn (2) is the position of the moving parcel (and has to be carried along separately). The latter position is, of course, a result of the fluid flow rather than our choice (aside from the initial or starting position).

The temporal resolution of the tracking measurement, δt, and the size of the control volume sampled by the current meter transducer may be important parameters. Specifically, if the control volume is too large, then fluid parcels may change their direction or speed during the time that they are interacting with the transducer. The resulting velocity mesasurement would then be some kind of space and time average. Such an average can be quite useful insofar as it retains the information of interest while reducing the total volume of information that has to be recorded. Here we would rather avoid this possibility, so let's imagine that we can make the duration of the tracking measurement and the size of the control volume as small as need be so that the velocity of parcels is unchanging during the measurement process. Once we make measurements on this microscale (which can be very small indeed), our parcel-tracking and current meter measurement techniques should converge to the same result for fluid velocity at a given position and time. If the measurements differed then we would probably infer a calibration error of the current meter transducer, since the definition of fluid velocity given by Eqn 2 is what we intend the current meter to measure (and the estimation of fluid velocity via particle tracking is much more direct than the operation of a transducer).

Now that we have learned (or imagined) how to make a fluid velocity measurement, we can begin to think about surveying the entire domain in order to construct a representation of the complete fluid flow. Clearly, this will require additional measurements and it will also require an important decision regarding the sampling strategy. Should we make these additional measurements by tracking a large number of fluid parcels as they wander throughout the domain, or, should we deploy additional current meters and measure the fluid velocity at many additional sites? In principle, either approach could suffice to define the flow. Nevertheless, the measurements themselves and the analysis needed to understand these measurements would be quite different, as we will see in examples below. In actual practice, of course, our choice of a sampling strategy for an experiment will hinge upon purely practical matters - the availability of measurement systems, numerical codes, and theory - that are essential tools to carry out the survey and complete the analysis.

The velocity of a moving fluid parcel (2) is often referred to as the 'Lagrangian' velocity as we have done here, and the corresponding velocity field is less commonly referred to as the 'Eulerian' velocity (less common because 'Eulerian' is almost always understood to be the default). This usage appears to doubly wrong, both historically, Euler developed both representations (Lamb, 1938), and by its implication that there are two kinds of fluid velocity. Nevertheless, it is a usage so thoroughly ingrained into the subject that we will not buck the tide. It is, however, essential that we understand that a Lagrangian velocity measured or computed by tracking a parcel as it goes through a given y and at a given t is exactly the same thing as the Eulerian velocity measured or computed at that same y and t, thus (back to R^1)

$$V_L(\xi_o, t) = V_E(y, t) \quad \text{if} \quad y = \xi(\xi_o, t). \tag{9}$$

This rather arid-looking formula is nothing more or less than a mathematical statement of fluid velocity that we defined on physical grounds in the opening paragraphs. It's meaning may be better stated in words - *there is a unique fluid velocity* - which can be described from either a Lagrangian or an Eulerian perspective.

We can always attempt to generate the Eulerian velocity field from Lagrangian data by the analysis procedure of interpolating or mapping the (usually) irregularly sampled Lagrangian data $V(\xi_o, t)$ on to a spatial grid. To know where to assign the velocity we will also have to know the position, $y = \xi(\xi_o, t)$. In the example considered here we have the huge advantage of knowing all the parcel trajectories via (5) and so we can so make the transformation Lagrangian to Eulerian explicitly. Formally, the task is to eliminate all reference to the parcel initial position, ξ_o, in favor of the position $y = \xi$.
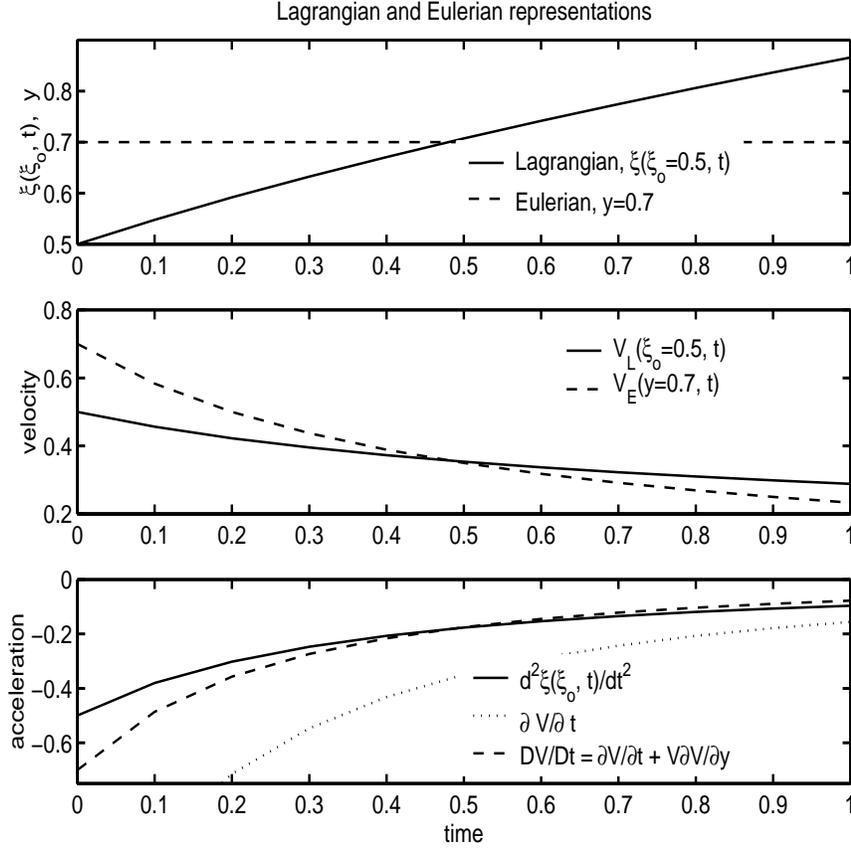
Figure 2: Lagrangian and Eulerian representations of the one-dimensional, time-dependent flow defined by (5). (Upper) The trajectory $\xi(\xi_o = 0.5, t)$ and the Eulerian observation site $y = 0.7$. (Middle) The Lagrangian and Eulerian velocity at $\xi_o = 0.5$ and $y = 0.7$, respectively. Note that the parcel identified by $\xi_o = 0.5$ crosses the Eulerian observation position $y = 0.7$ at time $t = 0.48$ that was computed from (7) by setting $\xi = 0.7$ and $\xi_o = 0.5$. At that precise moment these Lagrangian and Eulerian velocities are exactly equal, but generally not otherwise. That this equality holds is at once trivial - a non-equality could only mean an error in the calculation - but also consistent with and illustrative of a fundamental tenet of kinematics. (Lower) Accelerations for the parcel and the position used just above. There are two ways to compute a time rate change of velocity at a fixed point; one of them, DV/Dt, is the counterpart of the Lagrangian acceleration (discussed in detail in the next section).

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This is readily accomplished since we can invert the trajectory (5) to find ξ_o,

$$\xi_o = y(1 + 2t)^{-1/2}. \tag{10}$$

where we have already substituted y for ξ. Substitution into (6) and a little rearrangement gives the velocity field for this flow

$$V(y, t) = y(1 + 2t)^{-1} \tag{11}$$

which is plotted in Figure (1) lower. This (Eulerian) velocity field looks a little like the Lagrangian velocity of moving parcels (cf, Figure (1) middle), but the spatial coordinates are qualitatively different - ξ_o in place of a fixed position, y - and so the comparision is a bit like comparing apples and oranges. Nevertheless, there are times and places where the two velocities are the same. By tracking a parcel around in this flow and by observing velocity at a fixed site (in Figure 2 we arbitrarily chose the parcel tagged by $\xi_o = 0.5$ and the observation site $y = 0.7$), we can verify that the Eulerian and the Lagrangian velocities are equal at a common y and time consistent with (9) (Figure 2, middle). Indeed, there should be an *exact* equality since there has been no approximation made in this transformation Lagrangian → Eulerian.

The accelerations (Figure 2, lower) are a little more involved. For now, suffice it to say that the derivative with respect to time of velocity at a fixed position, $\partial V/\partial t$, is generally *not* equal to the Lagrangian acceleration at the same time and place, but another kind of Eulerian time derivative, denoted by DV/Dt, is equal to the Lagrangian acceleration, a very important matter taken up in detail in section IV. (For now - can you figure out why the Eulerian velocity at $y = 0.5$ decreases more rapidly than does the Lagrangian velocity on the parcel having $\xi_o = 0.5$?)

Summary. The two sampling strategies described above correspond with what are widely termed the Lagrangian and Eulerian (coordinate) systems, both of which are widely used in the analysis of continuum mechanics. In the Lagrangian system we seek to observe (or compute) the moving position, temperature, etc. of fluid parcels that constitute a fluid flow. In the Eulerian system we seek to observe (or compute) the velocity, temperature, etc. at points that are fixed in space. The Lagrangian perspective is natural for many measurement techniques and for the derivation of the fundamental conservation theorems. On the other hand, almost all of the theory in fluid mechanics has been developed from the Eulerian perspective. It may be apparent from the preceeding that it is useful (necessary?) to understand both systems, though we will go much farther with the Eulerian. Sometimes measurements made in one

system need to be transformed to the other and this transformation problem makes a conevenient theme of this discussion.

Kinematics of Fluid Flow II:
Lines.

It may be apparent from the previous discussion that simply showing what a fluid flow looks like will be a significant task in cases where the domain is multi-dimensional and the flow is time-dependent. A variety of methods are commonly used to show the flow dependence upon one or more of the independent variables, and it is important to understand what specific aspects of a flow can be revealed by one means or another. This topic of flow representation can be seen as a continuation of the Lagrangian-Eulerian transformation problem considered above. In this section we will assume that we know the Eulerian velocity field, and that we may need to compute certain Lagrangian properties of the flow.

Pathlines (or Trajectories) One important example is the parcel trajectories, often called pathlines. In this section we will consider position and velocity in a two-dimensional space, R^2, and $\mathbf{x}$ and $\mathbf{V}$ indicate vector position and velocity.

We can compute parcel trajectories from the Eulerian velocity field via

$$\frac{d\mathbf{x}}{dt} = \mathbf{V}(\mathbf{x}, t) \tag{12}$$

provided we recognize that $\mathbf{x}$ on the right side is the moving (time-dependent) parcel position. The appropriate initial condition is just

$$\mathbf{x}(t = 0) = \xi_o. \tag{13}$$

Note that (12) is in the form of the velocity indentity, Eqn (9). In component form this may be written out

$$\frac{dx}{dt} = U(x, y, t); \quad \frac{dy}{dt} = V(x, y, t) \tag{14}$$

and with the ICs

$$x(t = t_0) = \xi_{xo}; \quad y(t = t_0) = \xi_{yo} \tag{15}$$

which makes clear that we have two first order ODEs. On first sight these trajectory equations (14) could be deceptive; as here written they are quite general and applicable

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to any fluid motion in R^2. Thus it should not be surprising if on most occasions they prove intractable by elementary methods. If U depends upon y or V, or if V depends upon x or U, then these are coupled equations that have to be solved simultaneously; if U or V are nonlinear then they are nonlinear equations. Either way their solution may have to be sought with numerical techniques. What is surprising about (14), even after several encounters, is that what can seem to be very simple velocity fields can yield remarkably complex and interesting trajectories (one example is in Part III).

There are several useful diagnostic quantities that we can illustrate with a very simple two-dimensional flow constructed by adding an x-component velocity to the y-component velocity of Part I;

$$\mathbf{V} = x\mathbf{e}_x + \frac{y}{1+2t}\mathbf{e}_y, \tag{16}$$

which is plotted for two times in Figure 3. The component equations are then

$$\frac{dx}{dt} = x; \quad \frac{dy}{dt} = \frac{y}{1+2t}, \tag{17}$$

and with ICs as above. The dependent variables are uncoupled, and moreover, within each component equation the independent variables can be readily separated,

$$\frac{dx}{x} = dt; \quad \frac{dy}{y} = \frac{dt}{(1+2t)}.$$

These can then be integrated over the limits ξ_{xo} to x (ξ_{yo} to y) and t_0 to t to yield the trajectory

$$x(t; \xi_{xo}, t_0) = \xi_{xo} exp(t-t_0); \quad y(t; \xi_{yo}, t_0) = \xi_{yo} \frac{(1+2t)^{1/2}}{(1+2t_0)^{1/2}}. \tag{18}$$

Notice that the y-component is just as before (5), except that we have retained the initial time as a parameter (we'll need it below). Trajectories staring from a few different ξ_o and spanning two time periods are in Figure 4. In this case the trajectories are roughly in the direction of the flow as seen in Figure 3, and appear to bend over in time, consistent with the decreasing y-component of the velocity. In other cases, the trajectories can be a very intricate mix of the time and space dependence of the velocity field and thier form not anticipated in advance of the integration.

Streaklines. Another useful characterization of the history of parcel positions is the so-called streakline, which shows the positions, at a fixed time, of all of the parcels which at some earlier time passed through a given point. An example of this would be
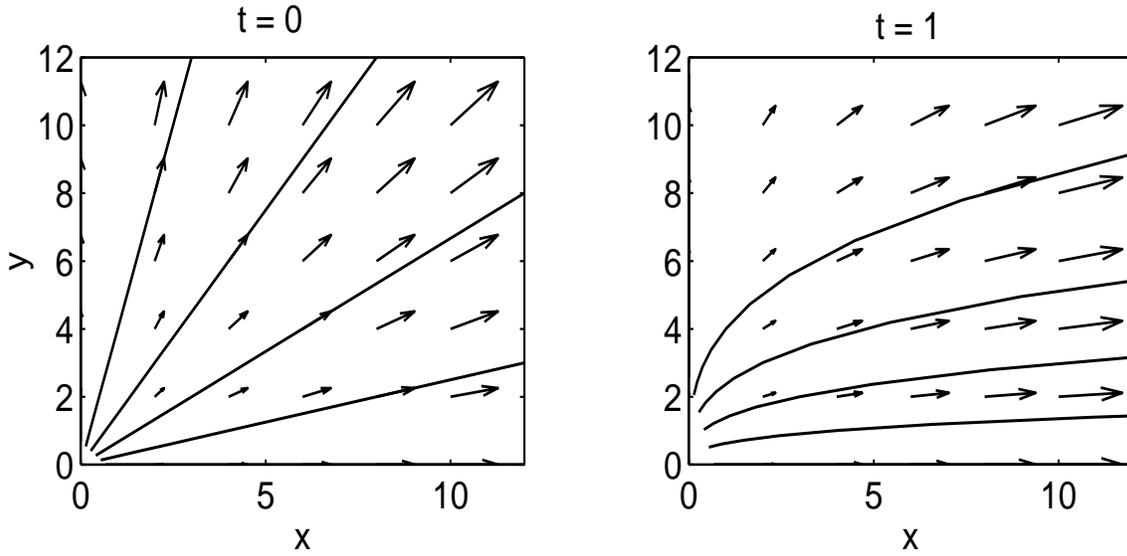
Figure 3: (left) Velocity field and streamlines for (17) at $t = 0$. (right) At $t = 1$. Notice that the velocity at a given point turns clockwise with time as the y-component of the velocity decreases with time.
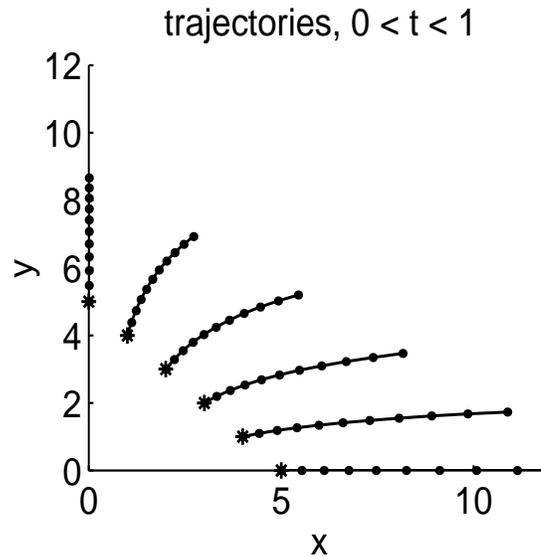

Figure 4: Trajectories of six parcels that were released into the flow (17) at the same time, $t_o = 0$, and tracked until $t = 1$. The sources are shown by asterisks. Dots along the trajectories are at time intervals of 0.1

the plume of smoke coming from a point source located at $\mathbf{x} = \mathbf{x}_p$ and recorded, say by a photograph taken at a time, $t = t_p$. The information needed to construct a streakline is contained within the trajectory, (18). To see this we will construct a streakline by releasing parcels one after the other from a fixed source. The first parcel is released at time $t_0 = 0$, and we let the trajectory run until $t = t_p$, the time we make the photograph. The only data point we retain from this trajectory is the position at time $t = t_p$, i.e., we record $\mathbf{x}(t_p; \mathbf{x}_p, t_0 = 0)$. A second parcel is released a little later, say at $t_0 = \frac{1}{4}$, and again we let the trajectory run until $t = t_p$, where we retain only the last point, $\mathbf{x}(t_p; \mathbf{x}_p, t_0 = \frac{1}{4})$. A third parcel is released at $t_0 = \frac{2}{4}$, and again we record it's position at $t = t_p$, $\mathbf{x}(t_p; \mathbf{x}_p, t_0 = \frac{2}{4})$. It appears, then, that a recipe for making streakline from a trajectory is that we treat the initial time, t_0, as a variable, while holding t constant at t_s, and also the initial position. Several such streaklines are in Figure 5. Notice that in this time-dependent flow, trajectories and streaklines are not parallel.

Streamlines. Still another useful 'line' is the streamline, a family of lines that are everywhere parallel to the velocity. Time is fixed, say at $t = t_f$, and thus streamlines portray the direction field of a velocity field, with no reference to parcels or trajectories or time-dependence of any sort. There is more than one way to construct a set of streamlines, but a method that lends itself to considerable generalization is to solve for the parametric representation of a curve, $\mathbf{X}(s)$ that is everwhere parallel to the velocity;

$$\frac{d\mathbf{X}}{ds} = \mathbf{V}(x, y; t_f) \tag{19}$$

or in components;

$$\frac{dX}{ds} = U(x, y; t_f); \quad \frac{dY}{ds} = V(x, y; t_f). \tag{20}$$

A suitable 'initial' condition is $X(s_0) = X_0$, etc. Notice that s is here a dummy variable; we could just as well have used any other symbol but s is conventional. $\mathbf{X}$ is the here the position of a point on a line, where just above $\mathbf{x}$ meant the position of a parcel. This reuse of symbols is certainly a risky practice for unwary readers, but it's also almost unavoidable. Given the velocity components (17), these equations are also readily integrated to yield a family of streamlines:

$$X = X_o exp(s - s_0); \quad Y = Y_o exp\left(\frac{s - s_0}{1 + 2t_f}\right), \tag{21}$$

and recall that t_f is the fixed time that we draw the streamlines. We are free to choose the integration constants so that a streamline will pass through the positions that we specify. There is no rule for choosing these positions; in Figure 3 we arbitrarily picked
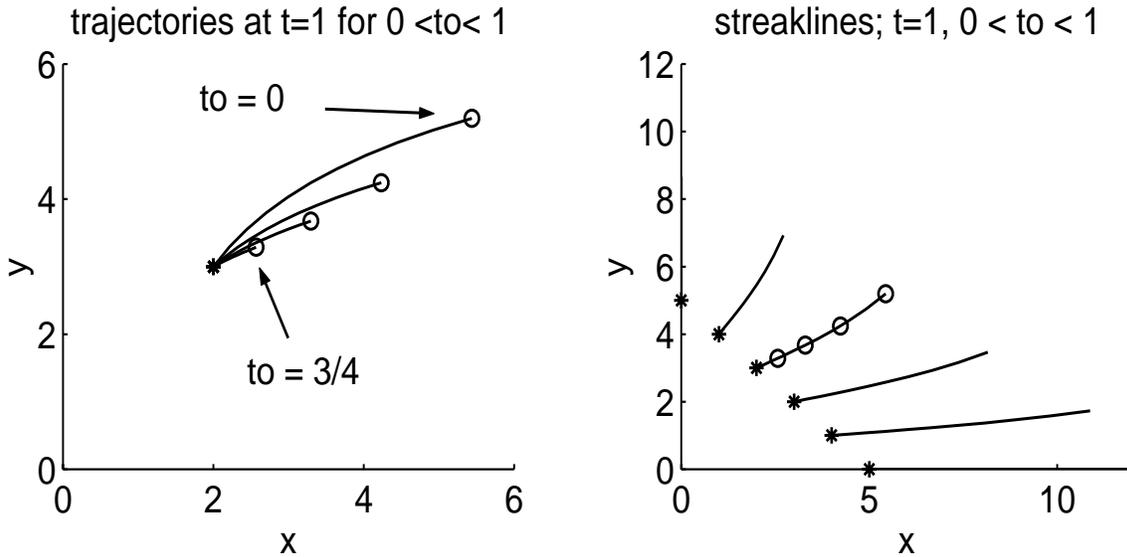
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Figure 5: (left) Trajectories of five parcels that were released from a common source, (x,y) = (2,3), and tracked until $t = 1$. The parcels were released at different initial times, $t_0 = 0$, 1/4, 2/4, 3/4, and 1. The latter trajectory has zero length. The end points of the trajectories are the open circles, the locus of which forms a streakline. (right) Streaklines from several different sources. These streaklines start at $t_0 = 0$ and the 'photograph' was taken at $t_p = 1$. Notice that these streaklines end at the endpoint of the trajectories of Figure 4 (they have that one point in common) but that streaklines generally have a different shape (different curvature) from the trajectories made over the same time range.

five positions and then let s vary over sufficient range to sweep through the domain. Other streamlines could be added if needed to help fill out the picture. No particular value is attached to a given streamline. In a later class we will consider the streamline's sophisticated cousin, the streamfunction, which is also parallel to velocity, but which has a value that is assigned in a way related to the speed of the flow.

Kinematics of Fluid Flow III:
Eulerian to Lagrangian Transformation by Approximate Methods; Stokes Drift.

The example that we have gone though above serves to show the purely formal steps required to transform from one reference frame to the other. An understanding of the formal steps is important, of course. However, the ease with which we could make the transformation in that case may be positively misleading. In actual practice, we will almost never have an explicit and invertible specification of trajectories for an entire domain (or in the approach (15) will not be able to separate variables) and so approximate methods are generally required to go from Eulerian velocity data to Lagrangian trajectories. The ODE system (15) can be solved by numerical methods even for very complex velocity fields or for fields known only on finite grids. If the velocity field can be written explicitly, though not necessarily integrated, then an approximate method, expansion in Taylor series, may be very useful.

The power and the limitations of the Taylor series method can be appreciated by analysis of a very simple flow, a steady, circular vortex in which the azimuthal current decays with distance r away from the center at a rate $U_\theta = C/2\pi r$. C is termed the circulation of the vortex, and is a measure of the vortex strength having units of speed*length. Here we will take $C = -2\pi$. This kind of vortex, often called a free vortex, is an idealization of the velocity distribution produced by the convergent flow into a drain, for example, and has some interesting properties that we will consider later on. For now it makes a convenient flow into which we can insert floats and current meters to investigate kinematics. It is fairly obvious that parcel trajectories around this vortex will be circular, and that a parcel will make a complete revolution in a time $2\pi r/(C/2\pi r) = (2\pi r)^2/C$.

Lets imagine that we do not know that the flow field is that of a free vortex, and that all we have are measurements of the velocity made at one fixed site (now in Cartesian coordinates) $(x, y) = (x_s, y_s) = (0, 1)$. The velocity observed at this site (an Eulerian velocity) would then be a steady, uni-directional flow to the right (x component only) at a speed $U_s = C/r = 1$. What, if anything, can be inferred about parcel trajectories from this data? If no other information was available, then we might make a first attempt at estimating a trajectory by integrating the Eulerian velocity in time as if it were the Lagrangian velocity. It is essential to understand that such a procedure is wrong, formally. But it is also fruitful to see the result, termed a progressive vector diagram, or PVD, as a lowest order approximation. The PVD in this case indicates a linear (pseudo-)trajectory, consistent with the uni-directional velocity measured at our fixed site (Figure 6). A PVD is a useful way to visualize a current meter record or a wind record in so far as it gives a direct measure of how much fluid has gone past the observation site. But the question here is to what extent does a PVD show *where* fluid

parcels have gone? That's a hard question to answer directly, but a related question - is there any condition or any flow under which we could we interpret a PVD as if it were a parcel trajectory? - leads us toward a useful analysis. One answer to the latter question is that this PVD (or any PVD) would be perfect *if* the flow were spatially uniform so that the Lagrangian velocity did not depend upon ξ_o, and the Eulerian velocity did not depend upon x or y. Measurements made anywhere would then be equal to measurements made anywhere else, including at the moving position of a parcel. A spatially uniform flow is a degenerate case, but helps us to see that the issue here is spatial variation of the flow.

If we consider the example of a steady vortex flow, then it would appear that the PVD is an acceptable trajectory estimate only for (pseudo-)displacements,

$$\delta \mathbf{X}_0 = \int \mathbf{V}(x_s, y_s) dt = U_s t \mathbf{e}_x + 0 \mathbf{e}_y \tag{22}$$

that are much less than the horizontal scale of the vortex. i.e., the horizontal distance over which the flow changes significantly (we will drop the pseudo- from here on). By inspection, the scale of this vortex is estimated to be the radius (at a given point), and so this condition could be written $\delta \mathbf{X}_0 << r$. For displacements greater than this, the velocity at the position of the parcel (the Lagrangian velocity that we should be integrating), begins to differ significantly from the velocity measured back at the fixed site. Once this discrepency is evident, the PVD fails (or will soon fail) to make a good approximation to a parcel trajectory.

We could begin to improve on this attempt at computing a trajectory from Eulerian data if we took account of the spatial variation of the velocity. To do this we might represent the velocity field in the vicinity of the observation point by expanding in a Taylor series, here for each component separately,

$$U(x, y) = U(x_s, y_s) + \frac{\partial U}{\partial x}(x_s, y_s)\delta X_0 + \frac{\partial U}{\partial y}(x_s, y_s)\delta Y_0 + HOT,$$

$$\tag{23}$$

$$V(x, y) = V(x_s, y_s) + \frac{\partial V}{\partial x}(x_s, y_s)\delta X_0 + \frac{\partial V}{\partial y}(x_s, y_s)\delta Y_0 + HOT.$$

The first derivatives of the velocity components are evaluated at the observation site, and the displacement components δX_0 and δY_0 are evaluated from the measured velocity ($\delta Y_0 = 0$ in this case, but carried along here for completeness). The PVD corresponds to the first term of the Taylor series, which we will call the zeroth order term, since the PVD does not depend upon displacement. The second term we will call
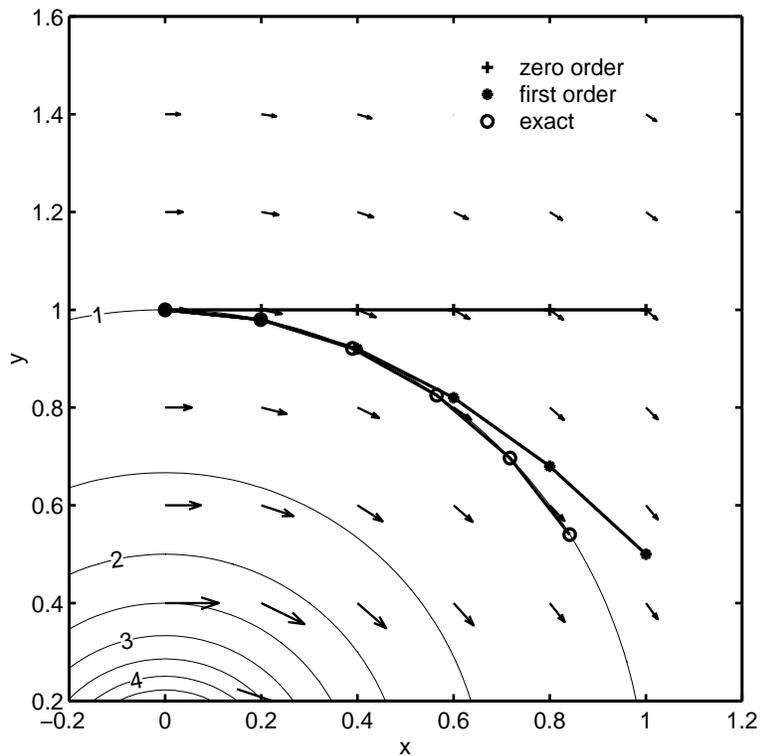
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Figure 6: Parcel tracking around a free vortex. Contour lines are of azimuthal speed with arrows indicating direction. Positions of a parcel launched at $(x, y) = (0, 1)$ are shown at times = 0, 0.2, 0.4, 0.6, 0.8 and 1.0. The exact trajectory was easily computed in cylindrical coordinates. The straight path is the zero order approximation, or PVD, in which the Eulerian velocity measured at the fixed observation site has been integrated as if it were the Lagrangian velocity of the parcel. Note that the PVD starts off well enough, but begins to accumulate a noticeable error as the displacement becomes even a small fraction of the radius. The first order trajectory makes use of the first spatial derivatives of the velocity components to account approximately for the spatial variation of the velocity. The first order trajectory is significantly better for the same displacement, but it too becomes rather inaccurate as the displacement becomes comparable to the radius.

the first order term since it has the first power of the displacement. The *HOT* are terms that are higher order in the displacement and derivatives. It is very convenient to use a vector and matrix notation to write equations like these in a format

$$\mathbf{V}(x, y) = \mathbf{V}(x_s, y_s) + \mathbf{G} \cdot \delta\mathbf{X}_0 + HOT, \tag{24}$$

where $\delta\mathbf{X}_0 = \mathbf{V}(x_s, y_s)t$ is defined by (22) and where

$$\mathbf{G} = \begin{pmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\[2mm] \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{pmatrix}$$

is the velocity gradient tensor. $\mathbf{G}$ recurs in the study of kinematics. For now $\mathbf{G}$ is a device to streamline notation; when we (matrix) multiply $\mathbf{G}$ into a displacement vector (which have to be column vectors), we get the velocity difference that corresponds to that displacement vector. It is easy to see that if we doubled (or halved) the length of the displacement vector we would get twice (or half) the velocity difference. Thus, multiplication by the velocity gradient tensor serves to make a linear transformation on the displacment vector. In general the result will be a velocity difference vector having a different direction from that of the displacement vector, and of course it has different dimensions (in the sense of mass, length, time) as well. The velocity gradient tensor is evaluated at the observation site, and has a particularly simple form in this case

$$\mathbf{G} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

where the dimensions are s^{-1} (you seldom see units attached to a tensor since most calculations are carried out with non-dimensional variables). The first order trajectory can then be readily computed by integrating (24)

$$\delta\mathbf{X}_1 = \delta\mathbf{X}_0 + \mathbf{G} \cdot \delta\mathbf{X}_0 t/2, \tag{25}$$

after noting that $\delta\mathbf{X}_0$ and $\mathbf{G}$ are independent of time since the velocity has been presumed to be steady. This first order trajectory is a considerable improvement upon the zeroth order PVD (Figure 6). Nevertheless, after sufficient time has passed and the displacement becomes comparable to the length scale of the flow (the radius) then this first order trajectory also accumulates a noticeable error. Adding an evaluation of the next *HOT* would delay the failure.

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Stokes Drift In a steady vortex flow of the type being considered above the flow will eventually carry a parcel long distances from its origin. As one result, approximation methods built around a truncated Taylor series expansion will necessarily have a limited validity. The method has better success applied to a wavelike motion, in which the parcel displacements are small, in a sense to be defined. As an important example, we can compute and understand the phenomenon of Stokes drift, which is one of the most important ways in which a wave motion and a mean flow can interact. We will analyze the parcel motions associated with a surface gravity wave having a surface displacement

$$\eta(x, t) = a\cos(kx - \omega t), \tag{26}$$

where a is the amplitude of the surface displacement, $k = 2\pi/\lambda$ is the wavenumber given the wavelength λ and ω is the wave angular frequency (ω and $k > 0$). The trigonometric function shows that this surface displacement moves rightward as a progressive wave having a phase speed $c = \omega/k$. Linear theory gives the velocity in directions parallel to the propagation and in the vertical associated with this wave

$$\mathbf{V} = a\omega e^{kz} \left(\begin{array}{c} \cos(kx - \omega t) \\ \sin(kx - \omega t) \end{array} \right), \tag{27}$$

where z is the depth, positive upwards from the surface. This velocity has an amplitude $a\omega = U_0$ at the surface, $z = 0$, and decays with depth exponentially on a scale $1/k$. This decay with depth is appropriate for a wave whose wavelength is considerably less than the water depth, a so-called deep water wave; for a wave with wavelength much greater than the water depth, a shallow water wave, the x component of the velocity is independent of depth.

Measured at a fixed point, the (Eulerian) velocity is a rotary current of amplitude $a\omega e^{kz}$ that turns clockwise with time at the angular frequency ω. The average of this orbital velocity over a wave period is exactly zero. From the known velocity components we can readily calculate the PVD-like parcel displacements,

$$\delta\mathbf{X}_0 = ae^{kz} \left(\begin{array}{c} -\sin(kx - \omega t) \\ \cos(kx - \omega t) \end{array} \right). \tag{28}$$

The PVD indicates that parcels move in a closed rotary motion with each wave passage and that the net motion is zero, consistent with the time-average of the velocity.

From the analysis of motion around a vortex we might have developed the insight that this PVD for a gravity wave would probably give a fairly accurate prediction for the
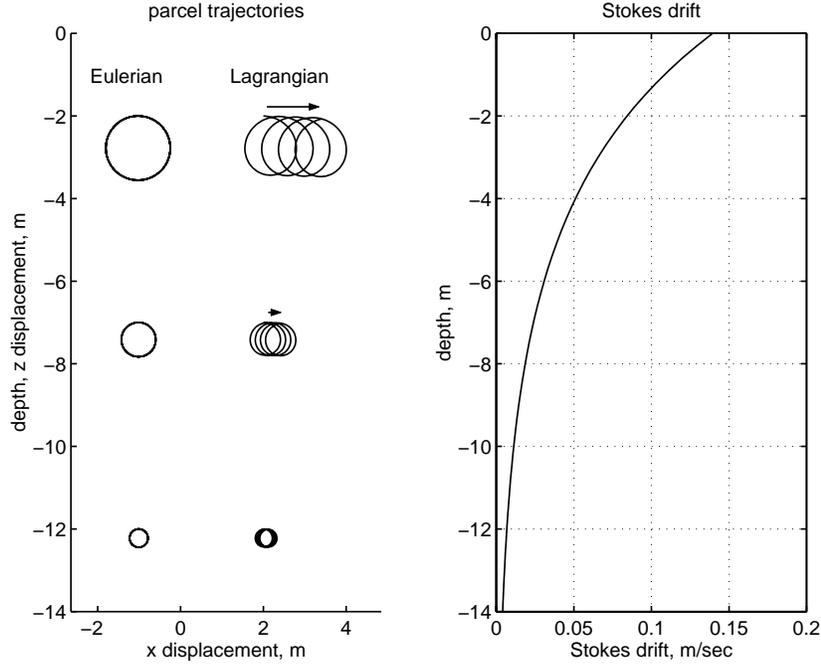
Figure 7: (left) Parcel trajectories underneath a surface gravity wave that is propagating from left to right. The Eulerian trajectories (or PVDs) are closed loops. The Lagrangian trajectories were computed by integrating numerically for four periods. Note that the Lagrangian trajectories are open loops so that there is a net drift of fluid parcels from left to right (shown as an arrow). (right) Stokes drift for this wave computed from (30).

actual parcel displacements provided that the parcel displacements were very much less than the scale over which the wave orbital velocity varies. In this case the scale is k in either direction, so that this condition is equivalent to requiring that the wave steepness, $ak = 2\pi/\lambda$ must be much less than 1. It is no accident that this is also the condition under which the linear solution gives an accurate waveform of the surface displacement (linear theory indicates a pure sinusoid, which we have assumed with (26)).

If we are dubious of this zero order approximation of trajectories then we may want to calculate the first order velocity via (24), and neglecting the HOT. The velocity gradient tensor for this wave is just

$$\mathbf{G} = a\omega k e^{kz} \begin{pmatrix} -\sin(kx - \omega t) & \cos(kx - \omega t) \\ \cos(kx - \omega t) & \sin(kx - \omega t) \end{pmatrix}.$$

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Matrix multiplying this into the zeroth order displacement gives the first order velocity,

$$\mathbf{V}_1(z) = a^2 \omega k e^{2kz} \begin{pmatrix} \sin^2(kx - \omega t) + \cos^2(kx - \omega t) \\ 0 \end{pmatrix}, \tag{29}$$

which could be termed the quasi-Lagrangian velocity (almost a Lagrangian velocity). We are especially interested in the possibility of a net or time-averaged motion, which can be readily computed by averaging over one wave period,

$$\overline{\mathbf{V}}_1(z) = U_o a k e^{2kz} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{30}$$

where $U_o = a\omega$ is the amplitude of the orbital velocity at the surface. Quite unlike the PVD, this indicates that fluid parcels have a substantial net motion, often called Stokes drift or mass transport velocity. The Stokes drift is a fraction $2\pi \frac{a}{\lambda} e^{kz}$ of the orbital wave motion. For example, for a wave having an amplitude of $a = 1$ m, and wavelength of 50 m, the orbital motion is about 1.11 m s^{-1} at the surface, and the surface Stokes drift is about 0.14 m s^{-1}. The Stokes drift decreases rapidly with depth.

It is easy to show that the vertically integrated Stokes drift,

$$M = \int_{-\infty}^{0} \overline{\mathbf{V}}_1(z) dz = U_0 a / 2$$

is related to the integrated kinetic energy density of the wave orbital velocity, K, and the phase speed, c, by the relation

$$M = \frac{2K}{c}$$

which is common to many other kinds of waves. Thus, the Stokes drift turns out to be much more than a residual effect of switching from an Eulerian to a Lagrangian coordinate system. Stokes drift represents the mass flux (momentum) associated with the waves, and is one of the most important means by which a wavelike motion and a mean flow can interact to produce a new velocity field. It is notable that all of the information about the Stokes drift was present in the Eulerian velocity field (27). However, to reveal this important phenomenon, we had to carry out an analysis that was consistent with the Lagrangian character of parcel motion.

Summary of Lagrangian-Eulerian Representations (Parts I, II and III). A key to understanding kinematics is to understand that a give fluid flow can be fully described using either a Lagrangian or an Eulerian representation. We have seen by way of a simple example that is possible to shift back and forth from Lagrangian to Eulerian representations provided that we have either

(1) a complete knowledge of all parcel trajectories, or,

(2) the complete velocity field at all relevant times.

In section I we first presumed (1), which happens only in the special world of homework problems. However, in a numerical model calculation we probably will have (2), and thus can compute parcel trajectories on demand. Numerical issues and diffusion (numerical and physical) will complicate the process and to some degree the result. Nevertheless, the procedure is straightforward in principle and is a common step in the diagnostic study of complex flows computed in an Eulerian frame. Approximate methods may be usefully employed in the analysis of some flows, an important example being the time-mean drift of parcels in a field of surface gravity waves. The Eulerian mean is zero, on linear theory, while the Lagrangian mean flow is substantial. In a general wave field, the relation between the three kinds of mean flows may be written

$$\mathbf{Lagrange} = \mathbf{Stokes} + \mathbf{Euler},$$

where the Eulerian mean is an average over time and space (locally). When the Stokes drift is appreciable, the Eulerian and Lagrangian mean flows may be in opposite directions.

Kinematics of Fluid Flow IV:
The Material Derivative

We noted in the first section that the application of Newton's laws of motion to a fluid is simplest in the Lagrangian coordinate system wherein it has the form of (solid) particle mechanics. The reason, worth repeating, is that in the Lagrangian system we are tracking specific parcels and momentum balance, for example, applies to a specific particle or fluid parcel and not to positions in space. In the Eulerian system the partial derivative with respect to time represents the rate of change (of velocity, say) at a point fixed in space, and will not equal the Lagrangian time rate change except in the somewhat degenerate case that there are no spatial variations of the flow. However, we have also emphasized that the velocity, temperature etc., of a fluid parcel is exactly the same thing as the velocity, temperature, etc. observed in an Eulerian frame at that same time and at the parcel position (if this statement sounds circular when we say it this way, then it is!). This suggests that we should be able to write the time rate of change following a parcel in terms of Eulerian (or field) variables, and this will turn out to be the key step in deriving equations suitable for modelling most fluid flows.

The Material Derivative in Field Form. To accomplish this transformation of a time derivative we have to write the time rate of change of a fluid variable, say temperature, T, in terms of $y(t)$, and as before eliminate ξ_o. Thus

$$\frac{dT(\xi_o, t)}{dt} = \frac{dT(y(t), t)}{dt}, \tag{31}$$

where we are assuming that the parcel trajectory (1) can be inverted to go from y to a unique ξ_o and t. Thus the fluid temperature written in field form depends upon both $y(t)$ and t, and so by the chain rule we find that the time rate of change is

$$\frac{dT(y(t), t)}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial t}. \tag{32}$$

The time rate of change of the parcel position is just the fluid velocity, $V = \frac{\partial y}{\partial t}$. It is customary to define the operator D/Dt and to write this expression in a form

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + V\frac{\partial T}{\partial y} \tag{33}$$

that will be used repeatedly in our development of fluid mechanics. D/Dt gives, entirely in field coordinates, the rate of change that would be observed following a parcel through the point y at time t. In Figure (2, lower) we showed an example, the time rate change of a velocity component following a parcel, $d^2y(t; \xi)/dt^2$, along with the field equivilent, DV/Dt defined above. As expected, these two accelerations are equal at a common position and time, but not otherwise.

The operator D/Dt goes by a number of different names, including the substantive derivative, the Stokes derivative and the material derivative (our choice), the profusion of names giving a clue to its importance. It is very often said to be the time derivative 'following the flow', in the sense that $\frac{\partial y}{\partial t}$ in (32) is the fluid velocity. This identification is central to this development, but it does not imply parcel tracking in the Lagrangian sense. In circumstances where confusion with the ordinary time derivative is not likely then the material derivative is often written as d/dt (and we will revert to this form once we are finished with kinematics and safely ensconced in an Eulerian frame).

The two terms that sum to make the material derivative are each important separately. The term $\partial/\partial t$ is the local time derivative, and is the rate of change observed at a fixed position. A steady flow is one that has $\partial/\partial t = 0$ throughout the domain. A steady flow need not be at rest, and may be subject to significant external forcing. The second term $V\frac{\partial}{\partial y}$ is called the convective or advective derivative and represents the rate of change of temperature, say, due to the flow of the parcel through a spatially

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varying temperature field. This term will vanish if the temperature is spatially uniform or if the fluid velocity is parallel to isolines of temperature.

An important because common balance among terms is that $D/Dt \approx 0$ because

$$\frac{\partial}{\partial t} \approx -V\frac{\partial}{\partial y}. \tag{34}$$

That is, it frequently happens that the material derivative of some property, let's say temperature, is a good deal smaller than the local rate of change of temperature because the local rate of change is due mainly to advection rather than to a (heat) source term. The assumption that this balance obtains is sometimes referred to as the 'frozen field' hypothesis, though the thing imagined to be frozen is in fact the spatial structure embedded within the moving fluid (and not frozen in space). When this spatial structure is carried past fixed points by the fluid flow there is then a local rate of change given by (34). Thus, the observation of a local rate of change of temperature, say, would not by itself be sufficient to infer the presence of a heat source unless the advection of temperature were known to be zero from some additional information.

In these examples we have assumed a one-dimensional domain to minimize the algebra. The same relations hold in a three-dimensional flow having Cartesian coordinates (x,y,z) and velocity (U,V,W), the material derivative then being

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} + W\frac{\partial}{\partial z}, \tag{35}$$

or using more compact vector notation,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{V} \cdot \boldsymbol{\nabla}. \tag{36}$$

The gradient operator expanded in Cartesian coordinates is the usual

$$\boldsymbol{\nabla} = \boldsymbol{e}_x\frac{\partial}{\partial x} + \boldsymbol{e}_y\frac{\partial}{\partial y} + \boldsymbol{e}_z\frac{\partial}{\partial z},$$

with $\boldsymbol{e}$ the unit vector. It is important to notice that the advective derivative term $\boldsymbol{V} \cdot \boldsymbol{\nabla}$ of (36) is treated like a scalar to be multiplied into the variable being differentiated. To emphasize this property, the advective derivative is often written as $(\boldsymbol{V} \cdot \boldsymbol{\nabla})$. Some care has to be taken when D/Dt is applied to vectors and expanded in other than Cartesian coordinates. In the Cartesian example we find that the D/Dt produces four terms for each component of the vector being differentiated, one term being the partial with respect to time, and three terms arising from the advective

term. However, when D/Dt is expanded in cylindrical polar coordinates we get five terms from the r and θ components with an additional advective term arising from the θ dependence of the r and θ unit vectors (and there is no corresponding cross-dependence of unit vectors in the Cartesian system). In spherical polar coordinates the D/Dt operator expands to six terms. It is sometimes convenient to represent the advective derivative of velocity by the following vector identity

$$(\mathbf{V} \cdot \nabla)\mathbf{V} = \frac{1}{2}\nabla(\mathbf{V}^2) + (\nabla \wedge \mathbf{V}) \wedge \mathbf{V}, \tag{37}$$

which is less likely to be misinterpreted. This form is especially useful in the idealized 'perfect fluid', wherein frictional effects vanish and so too does the curl of the velocity, also called the vorticity.

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Kinematics of Fluid Flow V:
Reynolds Transport Theorem

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We have indicated that the conservation principles of classical physics can be applied directly to a specific parcel or volume of fluid, but we have also indicated (without any supporting demonstration) that the development of a model of fluid flow is generally best done in a field coordinate system. Thus it will prove very useful to transform integrals taken over fluid volumes into their equivalent field form.

The Reynolds Transport Theorem. Consider the integral in R^1 of a scalar property, let's say temperature, T, over a moving 'volume' of fluid (the results are easily extended to R^3),

$$H(t) = \int_{y(t;\xi_1)}^{y(t;\xi_2)} T dy. \tag{38}$$

The limits on this integral are the positions of moving parcels. This integral is well-defined, and it is entirely sensible to ask for its time derivative, dH/dt, which we can expect will be proportional to an applied heat source (aside from divergence, which we will take up in the next part). It is far preferrable to specify heat sources in field coordinates rather than material coordinates, and thus the present development.

This integral looks a little exotic because the independent coordinates are material coordinates, but nevertheless it is no more than the sum of an integrand, in this case the temperature, T, times a differential length, dy, that happens to be embedded in a

moving fluid. To transform this integral to field variables we have to transform the time derivative of T (a scalar), which we have already done (part IV), and the time derivative of a material length, let's call it L to reduce the proliferation of d's;

$$L(t) = \int_{y(t;\xi_1)}^{y(t;\xi_2)} dy = y(t;\xi_2) - y(t;\xi_1). \tag{39}$$

L is just the distance between two moving parcels and its time derivative is then just the velocity difference

$$\frac{dL}{dt} = \frac{d}{dt}(y_2(t;\xi_2) - y_1(t;\xi_1)) = V_2 - V_1, \tag{40}$$

where V_1 is the fluid velocity at $y_1 = y(t;\xi_1)$. The velocity difference on the right side can be written $V_2 - V_1 = L\frac{\partial V}{\partial y}$ as L is made infinitesimal, and thus we conclude that the time derivative of an (infinitesimal) material length transforms to field coordinates as

$$\frac{d}{dt}dy_{material} = \frac{\partial V}{\partial y}dy_{field}, \tag{41}$$

where the velocity spatial derivative $\frac{\partial V}{\partial y}$ is in field coordinates and $dy_{field} = dy_{material}$ at the time that the transformation is made. Thus we have accounted for a possible change in the length of a material line. If instead of a line segment we transformed a differential volume, then the corresponding result would be

$$dvol^{-1}\frac{d}{dt}(dvol) = \nabla \cdot \boldsymbol{V}, \tag{42}$$

where

$$\nabla \cdot \boldsymbol{V} \equiv \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} \tag{43}$$

is the divergence of the fluid velocity.

With these results in hand, we are prepared to state the Reynolds Transport Theorem (RTT) which relates the time derivative over a material volume to the equivalent field quantities. For this one-dimensional case:

$$\frac{d}{dt}\int_{y(t;\xi_1)}^{y(t;\xi_2)} Tdy = \int_{y_1}^{y_2} (\frac{DT}{Dt} + T\frac{\partial V}{\partial y})dy, \tag{44}$$

where $y_1 = y(t;\xi_1)$, etc. at the time the transformation is made. Notice that the integral on the left side of the equation is over material coordinates, while the integral on the right side is over field coordinates (in case there could be any ambiguity the

type of coordinate will be written out). In three dimensions the RTT applied to a scalar T that has a body source Q is just

$$\frac{d}{dt} \iiint_{material} T \; dvol = \iiint_{field} (\frac{DT}{Dt} + T\nabla \cdot \boldsymbol{V}) \; dvol = \iiint_{field} Q \; dvol. \qquad (45)$$

The RTT is a purely kinematic relationship that applies to any fluid property, momentum, angular momentum, energy etc. The appropriate source will, of course, be different for each of these.

Mass Conservation. An important application of the RTT is to the mass of a moving, three-dimensional volume of fluid

$$M = \iiint_{material} \rho \; dvol. \qquad (46)$$

There is no body source of mass, and we can assert that M must remain exactly constant provided that we are following a specific volume of fluid (and ignoring the possibility of relativistic effects). Thus $dM/dt = 0$ with no approximation. (It is crucial to understand that we could make no such assertion for a volume that was fixed in space.) By means of the Reynolds Transport Theorem we can write this in field variables as

$$\frac{dM}{dt} = 0 = \iiint_{field} (\frac{D\rho}{Dt} + \rho\nabla \cdot \boldsymbol{V}) dvol, \qquad (47)$$

where the triple integral is over the spatial position of the volume. If this integral relation holds at all times and for all positions within a domain, and if the integrand is smooth (no discontinuities), then the integrand must vanish at all times and positions in that domain yielding the differential form of mass conservation,

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \boldsymbol{V} = 0. \qquad (48)$$

Thus if the density of a fluid parcel changes in time, say it increase, then we can say that there must have been an associated convergence (negative divergence) of the fluid velocity, $\nabla \cdot \boldsymbol{V} \leq 0$, and an associated decrease in the volume of the parcel. This holds regardless of the cause of the density change, whether due to a pressure variation or a heat source.[3]

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[3]This will sound a little strange to readers familiar with natural fluids such as the atmosphere or ocean. To see that this result holds generally but may nevertheless be irrelevent in many circumstances, consider a fixed mass of gas enclosed in a cylinder that is capped by a piston. In steady state the weight of the piston is supported by the pressure within the gas. Now imagine that the gas absorbs heat. Before we can calculate the new pressure and density of the gas we have to specify whether the piston

A second important application of the Reynolds Transport Theorem arises on consideration of momentum balance. The momentum of a moving, three-dimensional volume of fluid can be written

$$\boldsymbol{B} = \iiint_{material} \rho \boldsymbol{V} \; dvol = . \tag{49}$$

Because this is a material volume we can assert Newton's Second Law,

$$\frac{d\boldsymbol{B}}{dt} = \iiint_{material} \boldsymbol{F} \; dvol = \iiint_{field} \boldsymbol{F} \; dvol, \tag{50}$$

where $\boldsymbol{F}$ is the sum of all forces acting on the volume. (The second equality holds because the integral over the field volume is equal to the integral over the material volume assuming, as in all of this development, that the volumes coincide.) These forces could include body forces, such as gravity, that act throughout the volume, and forces that can be imagined to act on the surface of the volume, such as normal stress (pressure) and shear stress. By means of the Reynolds Transport Theorem and the mass conservation relation we can write the left side of (50) in field coordinates as

$$\iiint_{field} \rho \frac{D\boldsymbol{V}}{Dt} \; dvol = \iiint_{field} \boldsymbol{F} \; dvol \tag{51}$$

and the volume integrals are performed over the volume occupied instantaneously by the moving fluid. The volume considered here is arbitrary, and so the differential form of the momentum balance for a fluid continuum is

$$\frac{D\boldsymbol{V}}{Dt} = \frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = \boldsymbol{F}/\rho. \tag{52}$$

A term that might have been expected, $\frac{D\rho}{Dt}\boldsymbol{V}$, has dropped out by application of the mass conservation requirement. Thus the momentum of a fluid parcel (or marked fluid

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is free to move. Assuming that it is free to move, then the gas will expand just enough so that the pressure force on the cylinder balances the weight of the cylinder; the pressure within the gas will thus be unchanged, and the heating process could be termed be isobaric. In an isobaric heating process the gas will expand and consequently the density of the gas will decrease (since the mass of the gas is presumed unchanged). Now suppose that the piston is clamped so that volume of the cylinder enclosing the gas is held fixed. The pressure of the gas will increase as heat is absorbed, but the density will remain constant along with the volume and mass. This second kind of process might be termed isovolume. The atmosphere and ocean are not enclosed fluids, of course, so that heating and cooling processes are effectively isobaric and are accompanied by volume and density changes. These density changes are often computed from an equation of state with no reference at all to the divergence of the fluid velocity, despite (48). The approximation associated with ignoring the velocity divergence, known as the incompressibility assumption, is a valuable simplification that can be made with little error provided that the fluid velocity is much less than the speed of sound (we'll see this again later). On the other hand acoustic waves owe their entire existence to velocity divergence and associated pressure changes.

volume) can change only because of a change of velocity. To make (52) useful for prediction we have to specify the force, F, as well as suitable boundary and initial conditions, tasks that we will take up in succeeding sections.

Fluxes in Space. Mass conservation can be written in a so-called flux form by expanding the material derivative and collecting terms under the divergence operator,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0. \tag{53}$$

Budget equations that can be written in the form

$$\frac{\partial f}{\partial t} + \nabla \cdot \mathbf{g} = 0 \tag{54}$$

are said to be in 'conservative' form and have the following important property. Assume that the flux $\mathbf{g}$ is vanishingly small as x and y go to infinity, i.e., that the flow does not extend to infinity. Then by integrating both sides of the conservation law from $-\infty < x, y < +\infty$, and use of the divergence theorem on the right hand side we find that

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\partial f}{\partial t} dx dy &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \nabla \cdot \mathbf{g} \, dx dy \\
&= \oint \mathbf{g} \cdot \mathbf{n} \, ds \\
&= 0
\end{aligned} \tag{55}
$$

where in going from the second to the third line we have used that $\mathbf{g}$ vanishes at infinity and thus along the path of the line integral. The same thing would result if $\mathbf{g}$ vanished on the boundary of a domain, e.g., if the flux were due to fluid motion and the boundary was a closed container. If the flux across the boundary (or at infinity) vanishes, then we conclude that the spatial integral of f must be constant in time. The flux $\mathbf{g}$ may redistribute f within the domain so that f at a given point may be highly time-dependent. Nevertheless, $\mathbf{g}$ can not create or destroy the property f, on average (integrated over the domain as a whole). We would expect that to be true on purely physical grounds if f is fluid density and the flux $\mathbf{g}$ is density times fluid velocity; this can kind of conservation relation will hold for other properties, such as momentum and energy, provided that the momentum and energy budgets can be written in this flux or conservation form. Of course, if source terms are present in the budget for f then the integral of f would probably not be steady, and we should then call the corresponding equation an 'f budget' or 'f balance' rather than 'f conservation' (though this distinction is commonly ignored).

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If the volume (or areal integrals) in (58) are taken within a portion of a domain, and assuming a body source Q, then another important interpretation of the RTT is that

$$\frac{dF}{dt} = \int_a^b \int_c^d \frac{\partial f}{\partial t} dx dy \;\; = \;\; \int_a^b \int_c^d \nabla \cdot \mathbf{g} \; dx dy + \int_a^b \int_c^d Q dx dy$$
$$= \;\; \oint \mathbf{g} \cdot \mathbf{n} \; ds + \int_a^b \int_c^d Q dx dy \qquad (56)$$

where the time derivative can be moved inside the spatial integral over what are now presumed to be the fixed limits, a, b, etc., of a control volume. The total amount of f in this control volume, F, can thus change due to either a flux divergence within the control volume or a body source within the volume. Notice that the flux *per se* is not important; only the flux divergence appears in budget equations.

Summary of Transformation of Derivatives and Integrals, Parts IV and V. Derivatives and integrals can be transformed from Lagrangian to Eulerian coordinates, and the material derivative (the total time derivative following the fluid flow, written in field form) is an example of first importance. A clear understanding of what the material derivative means is an essential step in understanding the kinematics of fluid flow. Integrals and their time derivatives can also be transformed by way of the Reynolds Transport Theorem, and important examples include the mass conservation relation and the momentum balance. These two relations could be said to be the starting point for classical fluid mechanics.

References for Kinematics (I through V).

Aris, R., 1962. *Vectors, Tensors and the Basic Equations of Fluid Mechanics.*, Dover Pub., New York. [A rather advanced and mathematical source for fluid kinematics is Chapter 4 of this excellent text.]

Lamb, H., 1937. Hydrodynamics., Cambridge Univ. Press, 6th ed., 738 pp. [The classic tome on fluid dynamics before the age of numerical calculation.]

Longuet-Higgins, M. S., 1969. On the transport of mass by time-varying ocean currents. Deep-Sea Res., 16, 431-447. [The main source for the Stokes drift discussion of Section III.]

About This Document.

This essay is intended to be a supplement to the reading material assigned in 12.800, Fluid Dynamics of the Atmosphere and Ocean. It's sole purpose is educational.