12.540 Principles of the Global Positioning System
Lecture 12

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http://geoweb.mit.edu/~tah/12.540

Estimation

- Summary
  - Examine correlations
  - Process noise
    - White noise
    - Random walk
    - First-order Gauss Markov Processes
  - Kalman filters – Estimation in which the parameters to be estimated are changing with time

Correlations

- Statistical behavior in which random variables tend to behave in related fashions
- Correlations calculated from covariance matrix. Specifically, the parameter estimates from an estimation are typically correlated
- Any correlated group of random variables can be expressed as a linear combination of uncorrelated random variables by finding the eigenvectors (linear combinations) and eigenvalues (variances of uncorrelated random variables).

Eigenvectors and Eigenvalues

- The eigenvectors and values of a square matrix satisfy the equation $\mathbf{Ax}=\lambda\mathbf{x}$
- If A is symmetric and positive definite (covariance matrix) then all the eigenvectors are orthogonal and all the eigenvalues are positive.
- Any covariance matrix can be broken down into independent components made up of the eigenvectors and variances given by eigenvalues. One method of generating samples of any random process (ie., generate white noise samples with variances given by eigenvalues, and transform using a matrix made up of columns of eigenvectors.

Error ellipses

- One special case is error ellipses. Normally coordinates (say North and East) are correlated and we find a linear combinations of North and East that are uncorrelated. Given their covariance matrix we have:

$$\begin{bmatrix} \sigma_n^2 & \sigma_{ne} \\ \sigma_{ne} & \sigma_e^2 \end{bmatrix}$$ Covariance matrix;

Eigenvalues satisfy $\lambda^2 - (\sigma_n^2 + \sigma_e^2)\lambda + (\sigma_n^2\sigma_e^2 - \sigma_{ne}{}^2) = 0$

Eigenvectors: $\begin{bmatrix} \sigma_{ne} \\ \lambda_1 - \sigma_n^2 \end{bmatrix}$ and $\begin{bmatrix} \lambda_2 - \sigma_e^2 \\ \sigma_{ne} \end{bmatrix}$

Error ellipses

- These equations are often written explicitly as:

$$\left.\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix}\right\} = \frac{1}{2}\left(\sigma_n^2 + \sigma_e^2 \pm \sqrt{\left(\sigma_n^2 + \sigma_e^2\right)^2 - 4\left(\sigma_n^2\sigma_e^2 - \sigma_{ne}^2\right)}\right)$$

$$\tan 2\phi = \frac{2\sigma_{ne}}{\sigma_n^2 - \sigma_e^2} \quad \text{angle ellipse make to N axis}$$

- The size of the ellipse such that there is P (0-1) probability of being inside is

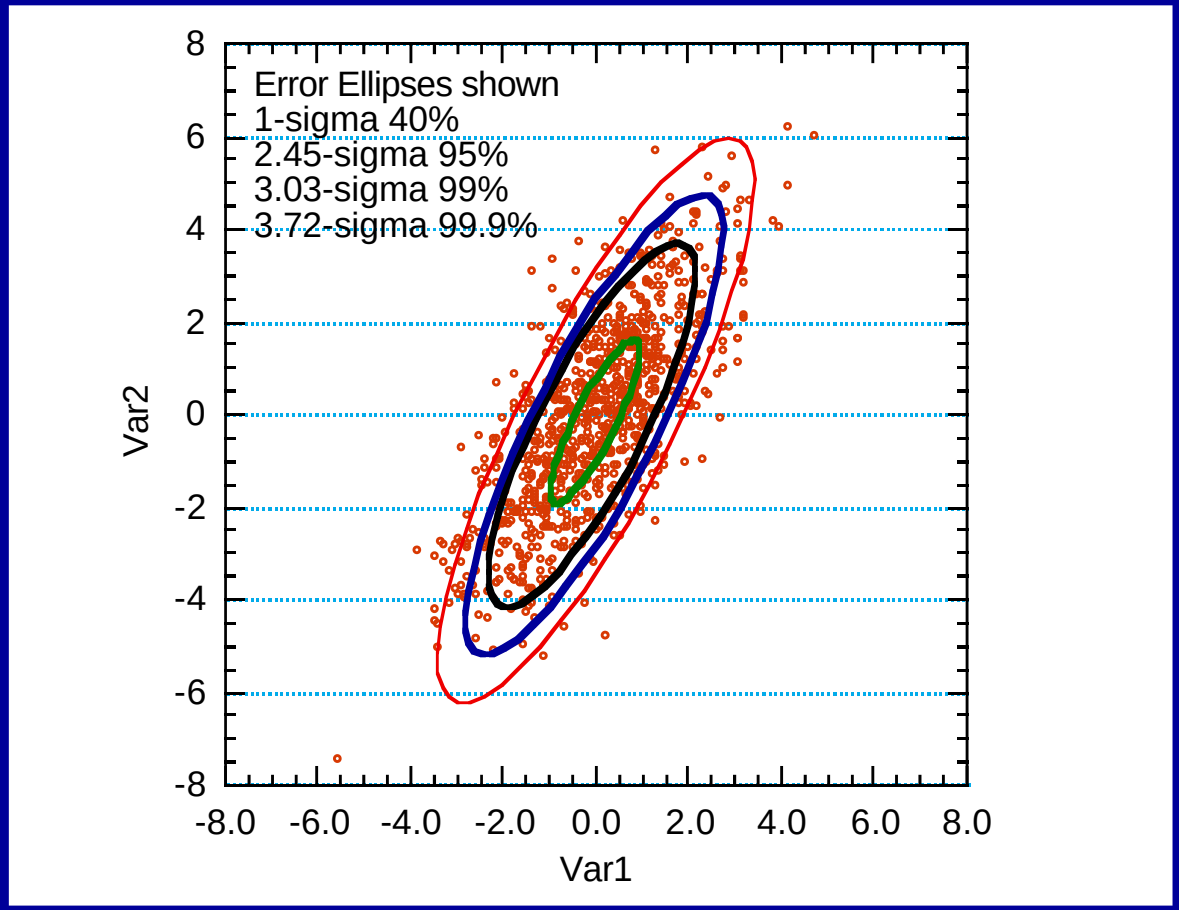
$$\rho = \sqrt{-2\ln(1-P)}$$

Error ellipses

- There is only 40% chance of being in 1-sigma error (compared to 68% of 1-sigma in one dimension)
- Commonly see 95% confidence ellipse which is 2.45-sigma (only 2-sigma in 1-D).
- Commonly used for GPS position and velocity results

Example of error ellipse

Covariance
2 2
2 4
Eigenvalues
0.87 and
3.66,
Angle -63^{o}



Process noise models

- In many estimation problems there are parameters that need to be estimated but whose values are not fixed (ie., they themselves are random processes in some way)
- Examples include for GPS
  - Clock behavior in the receivers and satellites
  - Atmospheric delay parameters
  - Earth orientation parameters
  - Station position behavior after earthquakes

Process noise models

- There are several ways to handle these types of variations:
  - Often, new observables can be formed that eliminate the random parameter (eg., clocks in GPS can be eliminated by differencing data)
  - A parametric model can be developed and the parameters of the model estimated (eg., piece-wise linear functions can be used to represent the variations in the atmospheric delays)
  - In some cases, the variations of the parameters are slow enough that over certain intervals of time, they can be considered constant or linear functions of time (eg., EOP are estimated daily)
  - In some case, variations are fast enough that the process can be treated as additional noise

Process noise models

- Characterization of noise processes
  - Firstly need samples of the process (often not easy to obtain)
  - Auto-correlation functions
  - Power spectral density functions
  - Allan variances (frequency standards)
  - Structure functions (atmospheric delays)
  - (see Herring, T. A., J. L. Davis, and I. I. Shapiro, Geodesy by radio interferometry: The application of Kalman filtering to the analysis of VLBI data, *J. Geophys. Res., 95*, 12561–12581, 1990.

Characteristics of random processes

- Stationary: Property that statistical properties do not depend on time

Autocorrelation $\varphi(t_1,t_2) = \iint_{x_1 x_2} x_1 x_2 f(x_1,t_1;x_2,t_2)dx_1 dx_2$

For stationary process only depends of $\tau = t_1 - t_2$

$$\varphi_{xx}(\tau) = \lim T \to \infty \frac{1}{2T}\int x(t)x(t+\tau)dt$$

PSD $\Phi_{xx}(\omega) = \int_{-\infty}^{\infty} \varphi_{xx}(\tau)e^{-i\omega\tau}d\tau$

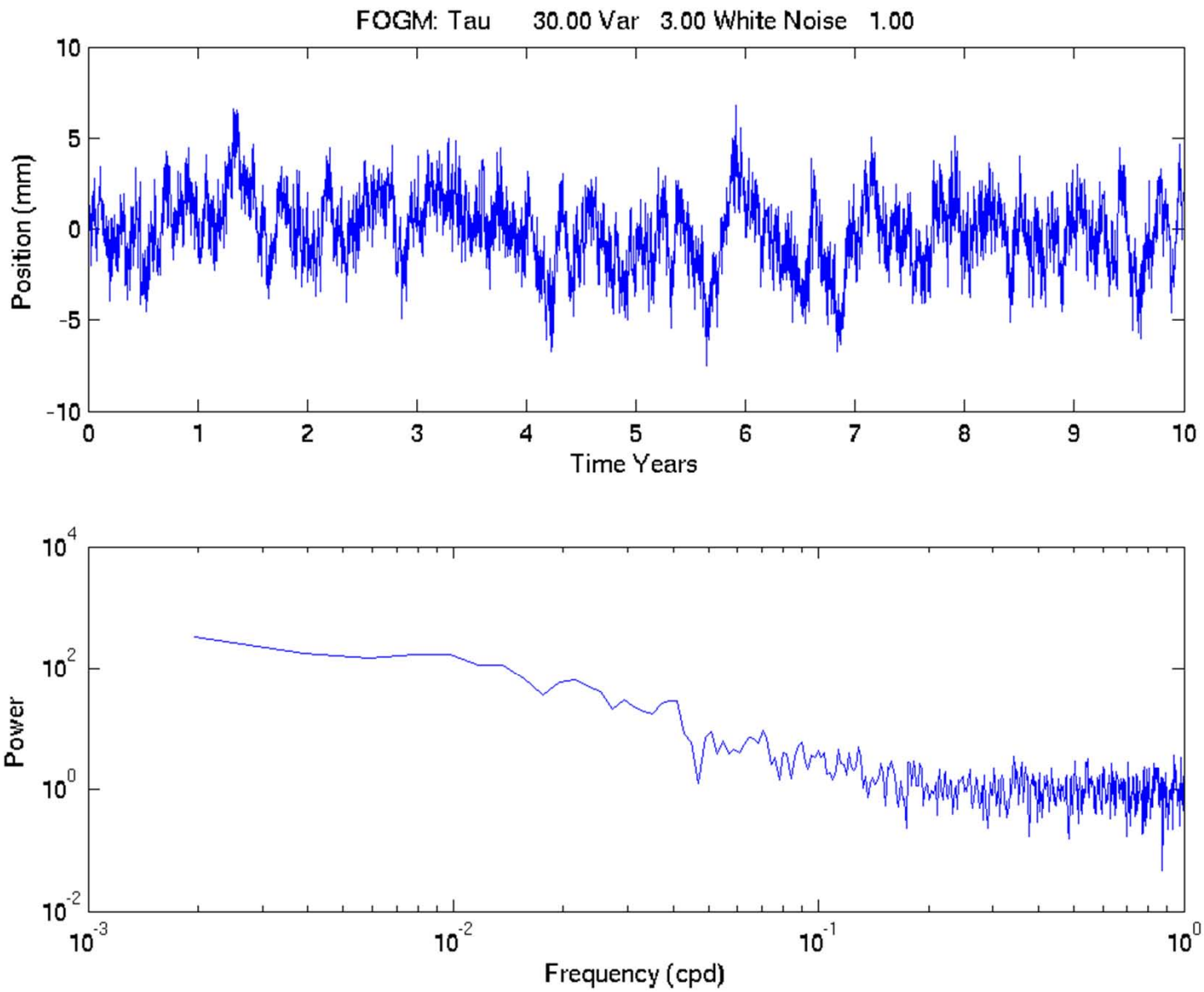
Specific common processes

- White-noise: Autocorrelation is Dirac-delta function; PSD is flat; integral of power under PSD is variance of process (true in general)
- First-order Gauss-Markov process (one of most common in Kalman filtering)

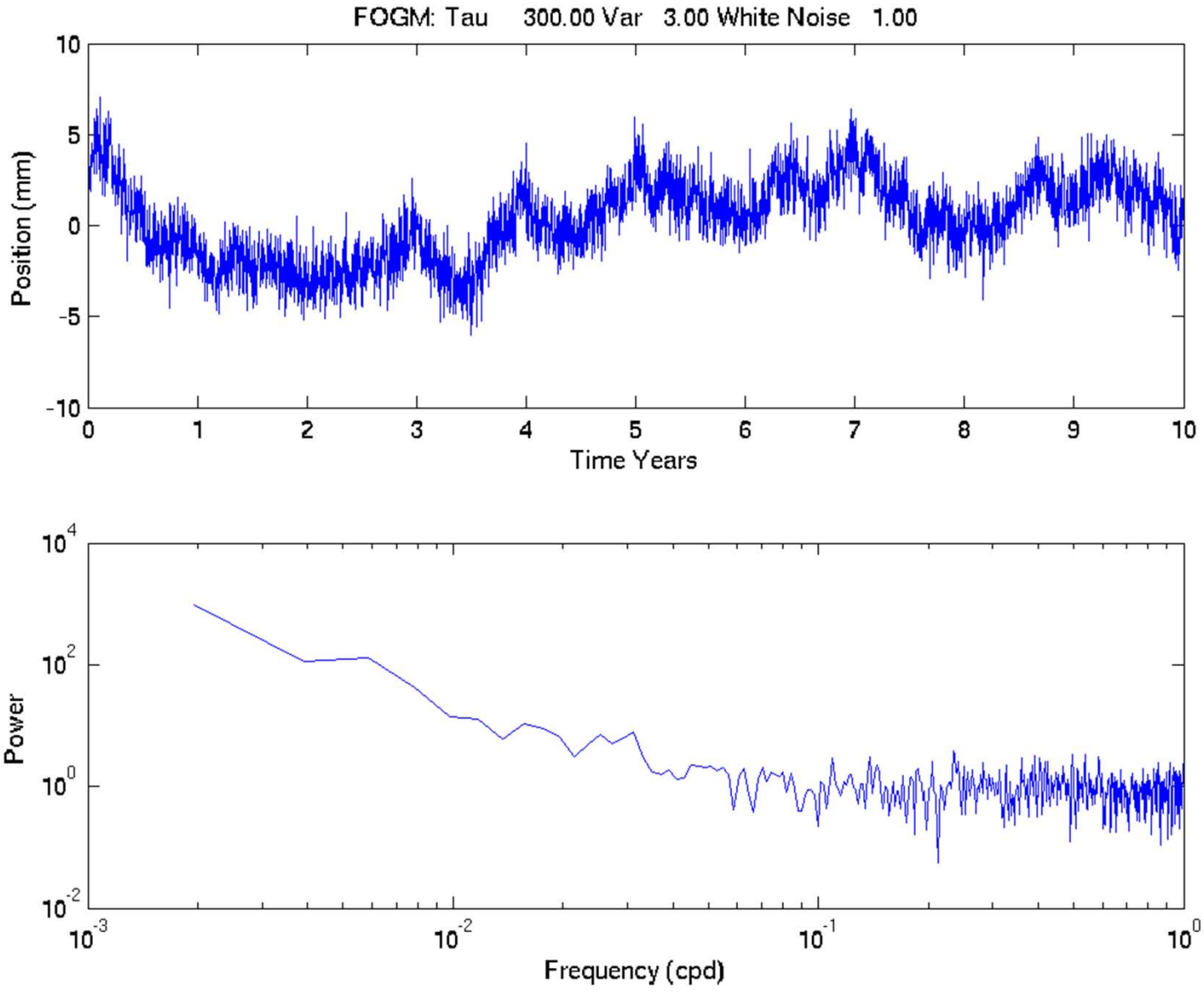
$$\varphi_{xx}(\tau) = \sigma^2 e^{-\beta|\tau|}$$

$$\Phi_{xx}(\omega) = \frac{2\beta\sigma^2}{\omega^2 + \beta^2} \qquad \frac{1}{\beta} \text{ is correlation time}$$

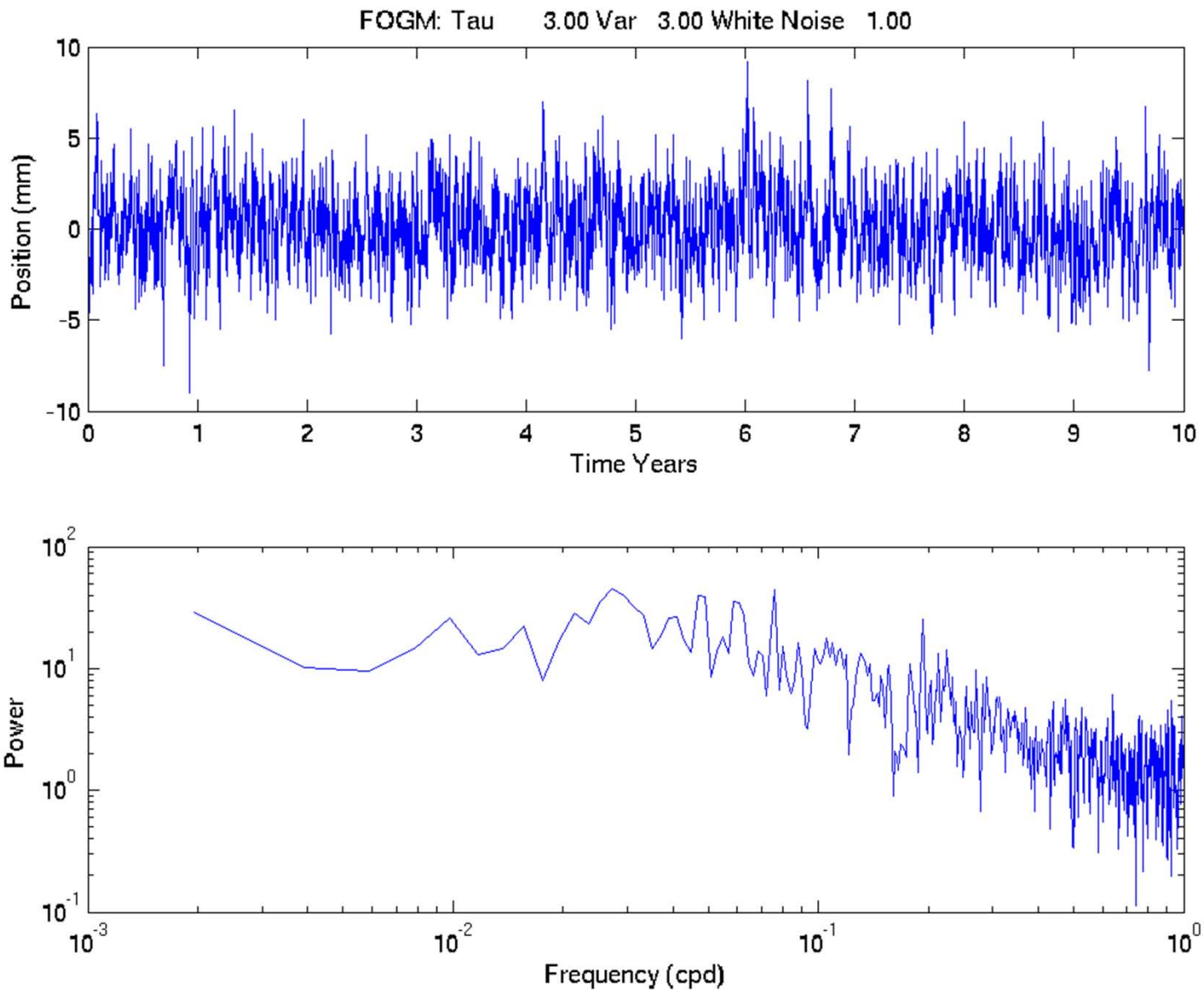
Example of FOGM process



Longer correlation time



Shorter correlation time



Other noise processes

- Other noise processes can be generated with various algorithms (all computational expensive)
  - Most General: Given any spectrum for the noise; inverse Fourier transform to generate covariance function (careful of aliasing); find eigenvectors and eigenvalues. Generate white noise with variances given be eigenvalues, multiple by eigenvector matrix. Resultant time series has spectral index properties.
  - For flicker noise, specific expressions exist for the covariance matrix (Ma et al., JGR, 1997)
  - Rodrigues et al., Chaos, Solutions and Fractals, 39, 2009 give expression of $f^{-\alpha}$ type noise.

Summary

- Examples show behavior of different correlation sequences (see fogm.m on class web page).
- The standard deviation of the rate of change estimates will be greatly effected by the correlations.
- Next class we examine, how we can make an estimator that will account for these correlations.
- Homework #2 in on the class web page (due April 18, 2012).

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Spring 2012