12.520 Lecture Notes 13

More Special Cases of Elasticity

Lecture 11 defined the constitutive equation for elasticity and explained how to reduce the number of terms using Lamé parameters. It also examined how to apply the elasticity equations in the special cases of uniaxial stress and uniaxial strain. This lecture explains how to apply the elasticity equations in two more special cases: plane stress and plane strain.

1. Plane Stress

Mathematics

Plane stress exists when only one principle stress is zero.

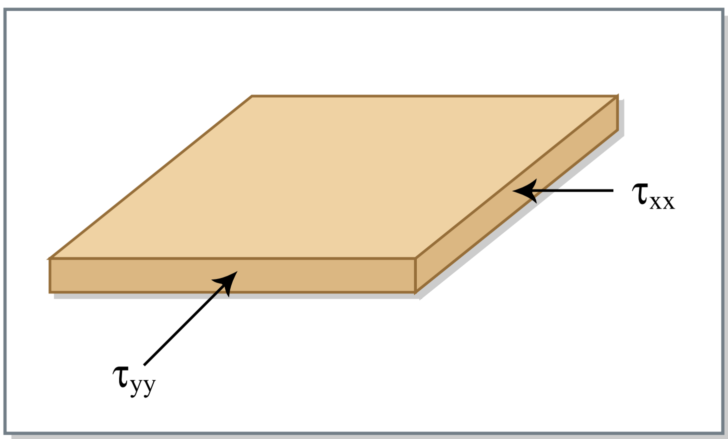

Figure 13.1
Figure by MIT OCW.

The stress tensor for the plane stress in the picture is given by

$$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \tau_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Substituting these values into the elasticity equations derived in the previous lectures yields:

$$\varepsilon_{xx} = \frac{1}{E}(\tau_{xx} - \upsilon\tau_{yy})$$

$$\varepsilon_{yy} = \frac{1}{E}(\tau_{yy} - \upsilon\tau_{xx})$$

$$\varepsilon_{zz} = -\frac{\upsilon}{E}(\tau_{yy} + \tau_{xx})$$

$$\varepsilon_{zy} = \frac{\tau_{xy}}{2\mu}$$

<u>Application</u>

Plane stress is useful because it often models the tectonic stress in the lithosphere. Under the conditions of plane stress, the total stress tensor is a combination of lithostatic stress and non-lithostatic plane stress:

Lithostatic stress	**Non-lithostatic stress**	**Total stress**
$\tau_{zz} = -\rho g z$	$\tau_{zz} = 0$	$\tau_{zz} = -\rho g z$
$\tau_{xx} = -\rho g z$	$\tau_{zz} = \Delta\tau_{xx}$	$\tau_{zz} = -\rho g z + \Delta\tau_{xx}$
$\tau_{yy} = -\rho g z$	$\tau_{yy} = \Delta\tau_{yy}$	$\tau_{yy} = -\rho g z + \Delta\tau_{yy}$

Assuming for simplicity that $\Delta\tau_{xx} = \Delta\tau_{yy} = \sigma$, the elasticity equations become:

$$\varepsilon_{xx} = \frac{\sigma}{E}(1-\upsilon)$$

$$\varepsilon_{yy} = \frac{\sigma}{E}(1-\upsilon)$$

$$\varepsilon_{zz} = -\frac{2\upsilon\sigma}{E}$$

The cubical dilation θ of the plate is given by

$$\theta = \varepsilon_{ii} = \frac{2\sigma(1-2\upsilon)}{E}$$

2. Plane Strain

Deriving the elasticity equations for the case of plane strain is very similar to deriving them for plane stress.

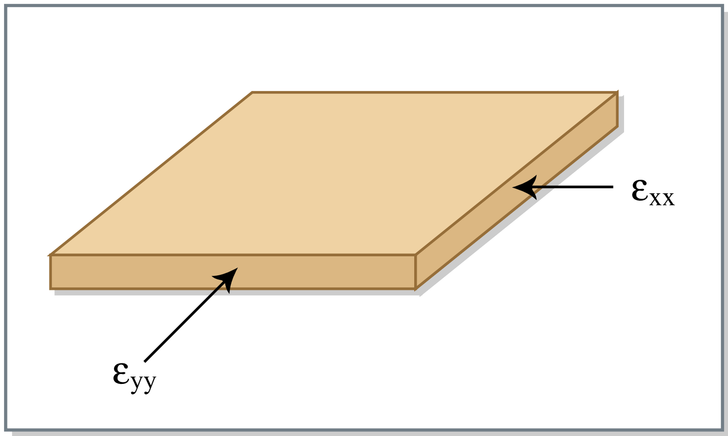

Figure 13.2
Figure by MIT OCW.

The strain tensor for the state of strain in the picture is given by

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{xy} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Substituting these values into the elasticity equations yields:

$$\tau_{xx} = (\lambda + 2G)\varepsilon_{xx} + \lambda\varepsilon_{yy}$$
$$\tau_{yy} = \lambda\varepsilon_{xx} + (\lambda + 2G)\varepsilon_{yy}$$
$$\tau_{zz} = \lambda(\varepsilon_{xx} + \varepsilon_{yy})$$