12.510 Introduction to Seismology
Spring 2008

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Feb. 27th 2008

Summary of the previous lecture:

In previous lectures, we learnt the stress tensor is given by:

$$\sigma_{ji} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \mathbf{T}^{(1)} \\ \mathbf{T}^{(2)} \\ \mathbf{T}^{(3)} \end{pmatrix} = \begin{pmatrix} T_1^{(1)} & T_2^{(1)} & T_3^{(1)} \\ T_1^{(2)} & T_2^{(2)} & T_3^{(2)} \\ T_1^{(3)} & T_2^{(3)} & T_3^{(3)} \end{pmatrix}. \qquad (1)$$

And that the traction vectors are given by:

$$T_i = \sigma_{1i}\, n_1 + \sigma_{2i}\, n_2 + \sigma_{3i}\, n_3 = \sigma_{ji}\, n_j \qquad (2)$$

Each component of the traction vector can be represented by:

$$\sigma_{ji} = T_i^{(j)} \qquad (3)$$

Where (j) indicates the surface and (i) represents the component.

In the absence of body forces, the stress tensor is symmetric $\sigma_{ij} = \sigma_{ji}$ so:

$$T_i = \sigma_{ji}\, n_j = \sigma_{ij}\, n_j \qquad (4)$$

The strain tensor is given by:

$$\varepsilon_{kl} = \frac{1}{2}\left\{ \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right\} \qquad (5)$$

The constitutive relationship for an elastic, isotropic medium is:

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} = \left((\lambda\delta_{ij}\delta_{kl}) + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \right)\varepsilon_{kl} \qquad (6)$$

Where c_{ijkl} is the stiffness tensor and λ, μ are the Lamé parameters, or “elastic constants”

This relationship can also be expressed as:

$$\sigma_{ij} = \underset{\substack{\uparrow \\ \text{Volume}}}{\lambda\delta_{ij}\theta} + \underset{\substack{\uparrow \\ \text{shear}}}{2\mu\varepsilon_{ij}} \qquad (7)$$

$$\theta = \nabla \cdot u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \qquad (8)$$

is the cubic dilation (or divergence of the displacement field)

The Equation of motion:

$$\sum F_i = ma_i \rightarrow \rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x_j}\left(\sigma_{ij}\right) + f_i = \partial_j \sigma_{ij} + f_i = \sigma_{ijk} + f_i \qquad \textbf{(9)}$$

In vector form, this gives: $\rho \ddot{u} = \nabla . \sigma + f$ **(10)**

Note: This is the equation of motion and should not be confused with the wave equations.

From the equation of motion, we derive the wave equations.

Often, we can approximate that the body forces, f **=0**

In low frequency seismology however, we cannot always make this approximation. For example, gravity is an important restoring force for the Earth's free oscillations. (When the earth expands, the restoring force is mostly due to its elasticity and due to gravity).

Also, when we are describing the seismic source, we introduce an 'equivalent' body force.

We can find the wave equation for the transmission of a displacement disturbance in a general elastic, homogeneous medium by substituting the constitutive relationship into the equation of motion:

$$\rho \ddot{u}_i = \rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x_j}\left[C_{ijkl} \frac{\partial uk}{\partial xl} \right] \underset{\substack{\uparrow \\ \textbf{Homogeneity}}}{=} C_{ijkl} \frac{\partial}{\partial x_j} \frac{\partial uk}{\partial xl} = C_{ijkl} u_{k,lj} \qquad \textbf{(11)}$$

Using: $C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$ **(12)**

This gives:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \left(\lambda + \mu\right) \frac{\partial}{\partial x_i} \frac{\partial u_k}{\partial x_k} + \mu \nabla^2 u_i \qquad \textbf{(13)}$$

Using vector notation, this equation of motion is written as:

$$\rho \ddot{u} = \left(\lambda + \mu\right) \nabla \left(\nabla . u\right) + \mu \nabla^2 u \qquad \textbf{(14)}$$

Now, we make use of the relationships:

$$\nabla . \left(\nabla \times a\right) = 0 \qquad \textbf{(15)}$$

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - (\nabla \times \nabla \times \mathbf{u}) \quad \textbf{(16)}$$

Note, if we are dealing with a rotation free field, equation (16) becomes:

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) \quad \textbf{(17)}$$

We must be careful, however to use the full relationship (16), if the field is not rotation free

Gravity is an example of a rotation-free field, because gravity is a conservative field

$$g = -\nabla U_{grav} \quad \textbf{(18)}$$

The magnetic field is an example of a field which is not rotation free.

Using the vector relationship (16), we can re-write the equation of motion as:

$$\rho \ddot{\mathbf{u}} = \underset{\substack{\uparrow \\ \text{dilatational}}}{(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u})} \; \underset{\substack{\uparrow \\ \text{rotational}}}{-\mu(\nabla \times \nabla \times \mathbf{u})} \quad \textbf{(19)}$$

This can be rearranged to give:

$$\ddot{\mathbf{u}} = \left(\frac{\lambda + 2\mu}{\rho}\right)\nabla(\nabla \cdot \mathbf{u}) - \left(\frac{\mu}{\rho}\right)\nabla \times (\nabla \times \mathbf{u}) \quad \textbf{(20)}$$

If we set: $\theta = \nabla \cdot \mathbf{u}$ **(21)**

And $\omega = \nabla \times \mathbf{u}$ **(22)**

We have:

$$\ddot{\mathbf{u}} = \alpha^2 \nabla\theta - \beta^2 \nabla \times \omega \quad \textbf{(23)}$$

This gives us expressions for the P and S-wave speeds:

The P-wave speed is given by: $\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}$ **(24)**

And the S-wave speed is given by: $\beta = \sqrt{\frac{\mu}{\rho}}$ **(25)**

We now want to go from this coupled equation to separate equations describing the motion of the P and S-waves.

Looking at the equation, we can see that an easy way to separate out the components is to take respectively the rotation and divergence of the left and right hand sides.

We can isolate the volume change by taking the divergence of both sides of the equation:

$$\alpha^2\nabla\cdot\nabla\theta-\beta^2\nabla\cdot(\nabla\times\omega)=\frac{\partial^2}{\partial t^2}(\nabla\cdot\mathbf{u})=\frac{\partial^2}{\partial t^2}\theta \quad \textbf{(26)}$$

Using: $\nabla\cdot(\nabla\times\omega)=0$ **(27)**

This simplifies to: $\alpha^2\nabla^2\theta=\ddot{\theta}$ **(28)**

We can separate out the shear part, by taking the curl of both sides of the equation:

$$\alpha^2\nabla\times\nabla\theta-\beta^2\nabla\times(\nabla\times\omega)=\frac{\partial^2}{\partial t^2}(\nabla\times\mathbf{u})=\frac{\partial^2}{\partial t^2}\omega \quad \textbf{(29)}$$

And using: $\nabla\times(\nabla\theta)=0$ **(30)**

this simplifies to: $\beta^2\nabla^2\omega=\ddot{\omega}$ **(31)**

So, we have decomposed the equation of motion into:

(1) A wave equation for dilational motion
(2) A wave equation for purely rotational motion, perpendicular to the direction of propogation

We can decompose the wave equation for shear waves into component form:

$$\beta^2\nabla^2\varpi_z=\frac{\partial^2\varpi_z}{\partial t^2} \quad \textbf{(32)}$$

$$\beta^2\nabla^2\varpi_y=\frac{\partial^2\varpi_y}{\partial t^2} \quad \textbf{(33)}$$

Equation 32 is the wave equation for the vertically-polarized shear wave S_v

Equation 33 is the wave equation for the horizontally-polarized shear wave S_H

<u>Decomposition of a vector field into Helmholtz potentials:</u>

Any vector field can be represented by a combination of the gradient of some scalar potential and the curl of a vector potential. These potentials are called 'Helmholtz potentials'

$\mathbf{u}=\nabla\phi+\nabla\times\Psi$ **(34)**

ϕ **= a scalar potential** $\nabla\times\phi=0$

$$\nabla.\psi=0$$

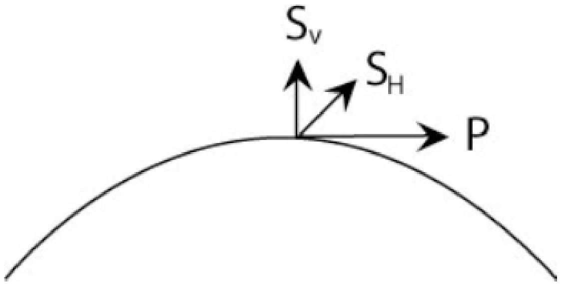
$\psi = \left(\psi_x,\psi_y\psi_z\right)$ **= a vector potential**

It follows directly that:

$$\frac{\partial\psi}{\partial y} = -\left(\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial z}\right) \qquad (35)$$

So, we can see that setting the divergence equal to zero, is equivalent to having only two independent components.

Figure 1:



Displacement v. Propagation direction

Locally, we can set up a new co-ordinate system that is associated with the propogation of the wave itself, as shown in the figure above.

Substituting for $\mathbf{u} = \nabla\phi + \nabla\times\Psi$ **into the equation:**

$$\rho\ddot{\mathbf{u}} = (\lambda + 2\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu(\nabla\times\nabla\times\mathbf{u})$$

And applying:

$$\nabla^2\mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - (\nabla\times\nabla\times\mathbf{u})$$

Gives:

$$\nabla\left\{\left(\frac{\lambda+2\mu}{\rho}\right)\nabla^2\phi - \ddot{\phi}\right\} + \nabla\times\left\{\left(\frac{\mu}{\rho}\right)\nabla^2\Psi - \ddot{\Psi}\right\} = 0 \qquad (36)$$

This equation can be satisfied by setting both brackets equal to zero.

This gives equations for:

The P-wave potential: (37)

$$\ddot{\phi} = \frac{\lambda+2\mu}{\rho}\nabla^2\phi$$

And the S-wave potential: (38)

$$\ddot{\Psi} = \left(\frac{\mu}{\rho}\right)\nabla^2\Psi$$

We can then reconstruct the solution $\mathbf{u} = \nabla\phi + \nabla \times \Psi$ using these Helmholtz potentials.

Expressing the curl of the vector potential as a matrix, we have:

$$\nabla \times \Psi = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \psi_x & \psi_y & \psi_z \\ \hat{x} & \hat{y} & \hat{z} \end{vmatrix} \qquad \textbf{(39)}$$

The displacement is a vector: $u = \left(u_x, u_y, u_z\right)$

Expressing the 3 components of u in terms of Helmholtz potentials, we have:

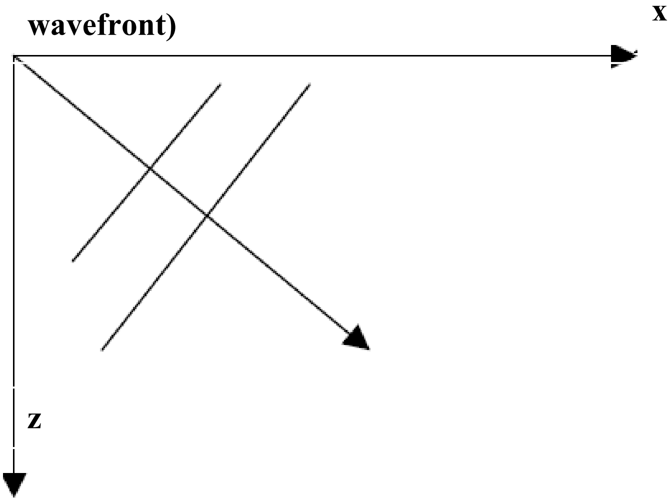
$$u_x = \frac{\partial\phi}{\partial x} + \left(\frac{\partial\psi_z}{\partial y} - \frac{\partial\psi_y}{\partial z}\right)$$
$$u_y = \frac{\partial\phi}{\partial y} - \left(\frac{\partial\psi_z}{\partial x} - \frac{\partial\psi_x}{\partial z}\right)$$
$$u_z = \frac{\partial\phi}{\partial z} + \left(\frac{\partial\psi_y}{\partial x} - \frac{\partial\psi_x}{\partial y}\right) \qquad \textbf{(40)}$$

In 2d, we consider propogation in the x-z plane:

Setting $\frac{\partial}{\partial y} = 0$ **gives:**

$$u_x = \frac{\partial\phi}{\partial x} - \frac{\partial\psi_y}{\partial z}$$
$$u_y = \frac{\partial\psi_x}{\partial z} - \frac{\partial\psi_z}{\partial x}$$
$$u_z = \frac{\partial\phi}{\partial z} + \frac{\partial\psi_y}{\partial x} \qquad \textbf{(41)}$$

Figure 2: Propogation of wave in x-z plane (arrow is used for they ray and solid line for the wavefront)



We can see that when we drop the derivatives with respect to y; u_y has only a Ψ dependence → pure shear (only s_H)

u_x and u_z both still have both ψ and ϕ dependences $\rightarrow\left(P-S_v\right)$ coupling

In 1D, the components of the displacement can be simplified even more:

We now have vertical motion.

$$u_x = -\frac{\partial \psi_y}{\partial z}, u_y = \frac{\partial \psi_x}{\partial z}, u_z = \frac{\partial \phi}{\partial z} \qquad (42)$$

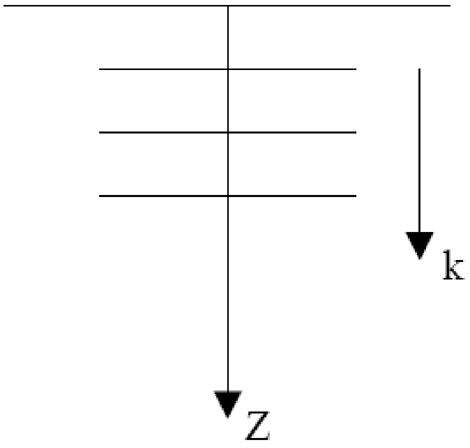
u_x → pure shear

u_y → pure shear

u_z → pure P-wave

S_H and S_v can't be distinguished because they are both in the horizontal plane.

Figure 3: Propogation of wave in 1d (arrow represents ray and solid line represents wavefront)



Note: The assumption of homogeneity

In deriving the equation of motion, we used the assumption of homogeneity:

$$\frac{\partial}{\partial x_j}\left(c_{ijkl}\varepsilon_{kl}\right) \underset{\uparrow}{=} c_{ijkl}\frac{\partial}{\partial x_j}\varepsilon_{kl}$$

Homogeneity

However, in many respects, we can use seismology to investigate the structure of the earth because it is not homogeneous.

On a global-scale, we can consider the earth to be made up of homogeneous layers.

If we take the limit of very high angular frequency $\left(\varpi \rightarrow \infty\right)$ the layers become very thin, and you effectively have a continuous medium.

This high-frequency approximation is used in 90-100% of exploration work.

Summary of the wave equations for the Helmholtz potentials:

$$\ddot{\phi} = \alpha^2 \nabla^2 \phi$$
$$\ddot{\Psi}_{SV} = \beta^2 \nabla^2 \Psi_{SV}$$
$$\ddot{\Psi}_{SH} = \beta^2 \nabla^2 \Psi_{SH} \quad \textbf{(43)}$$

Notes: Katie Atkinson, Feb 2008

Figures from notes of Patricia M Gregg (Feb 2005) Kang Hyeun Ji (Feb 2005)