Chapter 4

Tides

4.1 Tidal forcing

4.1.1 The "semi-diurnal" component

We need to consider how gravitational forces, due to the Sun or Moon, vary along the surface of the Earth. For simplicity in the following derivation, we shall focus on the Sun-Earth system (the Earth-Moon system produces the same result, but the analysis is alittle more complicated). We shall also neglect the inclination of the Earth's axis to its orbit, and consider only how the forcing varies along the equator; the geometry is shown in Fig. 4.1.

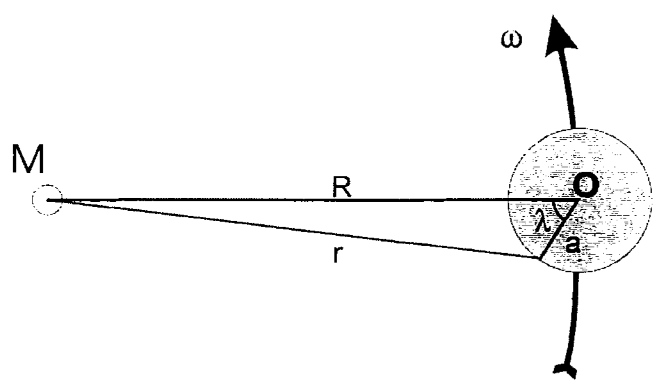

Figure 4.1: Gravitational tidal forcing.

Now, the gravitational potential at longitude λ (measured relative to the moving sub-solar or sub-lunar point on the Earth's surface, due to the tide-raising body (Sun or Moon) of mass M, is

$$\Phi_g = -\frac{GM}{r} = -\frac{GM}{\sqrt{R^2 - 2aR\cos\lambda + a^2}} ,$$

where R is the distance of the tide-raising body from the center of the Earth, and a the Earth's radius. Since $a/R \ll 1$, this can be approximated, correct to $O(a^2/R^2)$, as

$$\Phi_G \approx -\frac{GM}{R}\left(1 + \frac{a}{R}\cos\lambda - \frac{a^2}{2R^2}\left(1 - 3\cos^2\lambda\right)\right) .$$

Now, assuming that the center of the Earth's orbit coincides with the center of mass of M[1], the centrifugal potential is

$$\Phi_C = -\frac{1}{2}\omega^2 r^2 = -\frac{1}{2}\omega^2 R^2\left(1 - 2\frac{a}{R}\cos\lambda + \frac{a^2}{R^2}\right) ,$$

where ω is the angular velocity of the Earth in its orbit. Since the two components of force must balance at the Earth's center,

$$\omega^2 R = \frac{GM}{R^2} .$$

Therefore, the net variation of tidal potential around the equator is

$$\Phi_T = \Phi_G + \Phi_C = -\frac{3GM}{2R} - \frac{3GMa^2}{4R^3}\left(1 + \cos 2\lambda\right) . \tag{4.1}$$

The constant terms are, of course, irrelevant. The longitudinally-varying part describes a potential with wavenumber 2 around the globe: this is because the gravitational force decreases with distance, and the centrifugal force increases, so the former dominates at the subsolar (sublunar) point P, and the latter at Q, the antipodes of P (see Fig. 4.2).

[1]This is clearly a very poor approximation for the Earth-Moon system; however, the end result is the same.

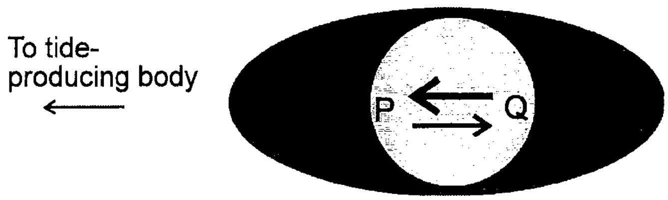

Figure 4.2: Illustrating the "semidiurnal" (wave 2) nature of the tidal potential.

Note that (because it is a *differential* measure of the gravitational field) the tidal forcing varies as R^{-3}.

The corresponding tidal force (per unit mass) is $-\nabla\Phi_T$; the horizontal component, along the Earth's surface, is the relevant one and this is just

$$F_T = -\frac{1}{a}\frac{\partial \Phi_T}{\partial \lambda} = -\frac{3GMa^2}{2R^3}\sin 2\lambda \ . \tag{4.2}$$

This is depicted in Fig. 4.3.

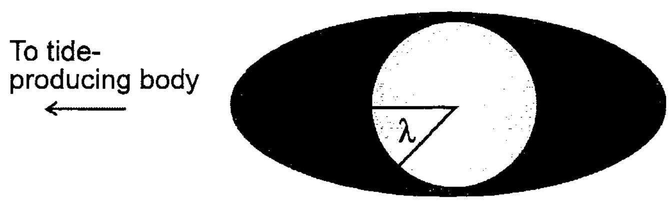

Figure 4.3: Tidal forces. The ellipse depicts the "equilibrium tide".

4.1.2 Lunar vs. solar forcing

Note that the magnitude of the tidal force depends on the properties of the tide-raising body as M/R^3. If the radius of the body is b, and its mean density ρ, then

$$\frac{M}{R^3} = \left(\frac{4\pi}{3}\right)\rho\left(\frac{b}{R}\right)^3 \ . \tag{4.3}$$

Now, because of the happy coincidence that the sun and moon subtend almost identical angles $\tan^{-1}(b/R)$ at the Earth, the ratio of their tidal forces is, by (4.3), approximately equal to their mean densities. As the lunar density exceeds that of the sun (by a ratio of about 2:1), lunar tidal forces are greater than solar, and the dominant tide in most places on the Earth is lunar semidiurnal (period of about 12hr 25min). The solar forcing is by no means negligible, however, which is why the tide goes through its monthly modulation from the high "spring" tides, when lunar and solar forcings are in phase, to the weaker "neap" tides, when they are out of phase.

4.1.3 The "diurnal" component

Since the inclination of the Earth's axis to the Earth-moon and Earth-sun lines is not zero, the tidal forces are not purely semidiurnal. As shown in Fig. 4.4,

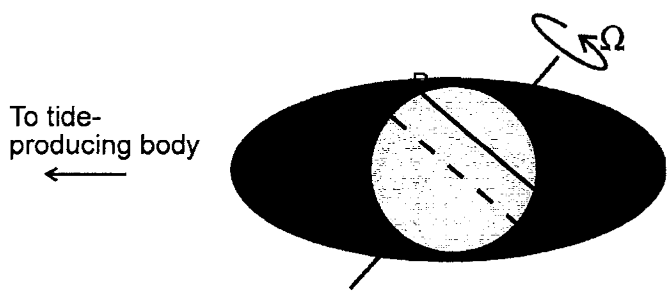

Figure 4.4: Illustrating the diurnal tidal component. Because the inclination of the Earth's axis is not zero, the high tide experienced at point P is weaker than that experienced 12 (lunar) hours later at point Q.

the tilt of the rotation axis relative to the potential surfaces introduces a diurnal asymmetry: the tidal potential maximum at Q is stronger than that at P. Thus, the tidal forcing has a diurnal, as well as semidiurnal, component. Note that the magnitude of this component will vary with period of a lunar month, as the orientation of the poles with respect to the Earth-moon line changes.

quantity	value	units
G	6.67×10^{-11}	$\mathrm{Nm^2kg^{-1}}$
M	7.30×10^{22}	kg
R	3.82×10^{8}	m
a	6.38×10^{6}	m
g	9.78	$\mathrm{ms^{-2}}$

Table 4.1: Data for the tidal calculation

4.2 Tides in the ocean

4.2.1 The "equilibrium tide"

The total gravitational potential around the Earth includes, of course, that due to the Earth's own gravity, $\Phi = gz$. If we consider the lunar forcing only, then if the Earth were not rotating, the surface of the ocean would, in equilibrium, coincide with a geopotential surface, on which, using (4.1),

$$-\frac{3GMa^2}{4R^3}\cos 2\lambda + gz = \text{ constant .}$$

This surface is shown schematically (and much exaggerated!) in Fig. 4.3. Since $\cos 2\lambda$ varies from -1 to 1, the extreme range (low to high tide) for the "equilibrium tide" is

$$Z_e = \frac{3GMa^2}{2gR^3} \tag{4.4}$$

Using the values from Table 4.1, we obtain the value[2] $Z_e = 0.545$m.

4.2.2 Tides in a global ocean

In reality, the tidal pattern remains fixed with respect to the Earth-moon axis, and so, relative to a point on the Earth's surface, it moves westward at a speed of $449\mathrm{ms}^{-1}$ at the equator. Now, a typical ocean depth is about $D = 5$km; since the wavelength of the tidal oscillation is $2\pi a/2 \simeq 2 \times 10^4$km, and this is very much larger than D, we can use shallow water theory to

[2] The actual value for the equilibrium ocean tide is about 0.7 of this. We have neglected to allow for the fact that the solid earth itself is tidally distorted, and that the solid-earth ocean system then produces a wave 2 modulation of the local gravity field.

deduce that the phase speed of free waves is $\sqrt{gD} \simeq 220\text{ms}^{-1}$. Therefore, we can hardly assume that the tide is steady, since it moves faster than free waves in the ocean. So the tide is *dynamic*—we need to consider the dynamic, rather than the static, response to the tidal forcing.

A complete analysis of tides on a global ocean (without interruption by continents) is a classic (if unrealistic) problem. Apart from needing to take account of the spherical geometry, we would also need to include the effects of the Earth's rotation, which is a significant factor for motions with periods of about 12 hrs. To avoid these complications and to get some (limited) insight into tidal motions, we consider the non-rotating problem of one-dimensional (E-W) motions in a narrow channel around a latitude circle (see Fig. 4.5).

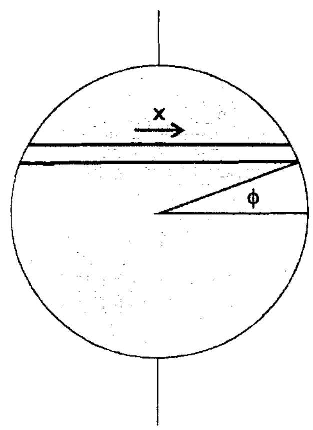

Figure 4.5: A narrow channel along a latitude circle, at latitude φ.

Since the channel is narrow, we can neglect curvature, and so use Cartesian coordinates, with x the coordinate in the longitudinal direction. The channel length is $L = 2\pi a \cos\varphi$. The relevant equations for this channel for the long tidal motions are just the shallow water equations, modified to include the tidal potential Φ_T:

$$\begin{aligned} \frac{du}{dt} &= -g\frac{\partial h}{\partial x} - \frac{\partial \Phi_T}{\partial x} \,; \\ \frac{dh}{dt} &= -h\frac{\partial u}{\partial x} \,. \end{aligned}$$

Since, from (4.4), we anticipate weak motions, we can reasonably linearize these equations about a state of uniform depth D and no motion; we then have

$$\begin{aligned}\frac{\partial u}{\partial t} &= -g\frac{\partial h}{\partial x} - \frac{\partial \Phi_T}{\partial x} \; ; \\ \frac{\partial h}{\partial t} &= -D\frac{\partial u}{\partial x} \; .\end{aligned} \tag{4.5}$$

Now, we know that the forcing has zonal wavenumber 2 and period 0.5 lunar day, so we write

$$\Phi_T = \Phi_0 \cos(2\lambda) = Re\left[\Phi_0 e^{ik(x-ct)}\right]$$

where, from (4.1), $\Phi_0 = 3GMa^2/(4R^3)$, $k = \pi/L = 2/(a\cos\varphi)$ and $kc = 2\pi/(0.5\tau)$, where τ is the lunar day. If we therefore seek solutions of the form

$$\begin{pmatrix} u \\ h \end{pmatrix} = Re\left[\begin{pmatrix} U \\ H \end{pmatrix} e^{ik(x-ct)}\right] , \tag{4.6}$$

then (4.5) give

$$\begin{aligned} cU &= gH + \Phi_0 \; ; \\ cH &= DU \; ; \end{aligned}$$

and so

$$H = \frac{D}{(c^2 - c_0^2)}\Phi_0 \; , \tag{4.7}$$

where $c_0 = \sqrt{gD}$ is the shallow water wave speed, as before.

In the limit $c \to 0$, this gives $H \to -D\Phi_0/c_0^2 = -\Phi_0/g$, i.e. the equilibrium tide, as we expect. Eq. (4.7) tells us:

(i) If $0 < c < c_0$, the tidal response is in phase with the equilibrium tide, and is larger;

(ii) If $c = c_0$, the response is **resonant**, since the system is being forced at its natural frequency;

(iii) If $c_0 < c < \sqrt{2}c_0$, the tidal response is larger than, **and out of phase with**, the equilibrium tide; and

(iv) If $c > \sqrt{2}c_0$, the response is smaller than, and out of phase with, the equilibrium tide.

Since, at the equator, the phase speed of the tidal forcing is 449ms^{-1}, while the wave speed is 220ms^{-1} for an ocean of 5km depth, resonance would occur for a channel at latitude $\arccos(220/449) = 60.7°$. We cannot of course take these results literally latitude-by-latitude, as the whole spherical system is coupled together. While this exercise gives us some idea of how the local dynamics are tending to behave, it does not describe the actual tides very well, as we shall now see.

4.2.3 Tides in ocean basins

Tidal observations are of course made in many locations, but most of these are coastal. To get a picture of what we think the global structure of tides looks like, we have to resort to output from numerical models. Such a picture of tides is shown in 4.6.

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The lines shown on Fig. 4.6 are of two types: **co-range** lines, which show the peak-to-peak amplitude (shown here with a contour interval of 0.25m, and labeled in meters), and **co-tidal** lines, which show the phase of the tide, expressed as the time of high water in "lunar hours" (about 1hr 2min) after the moon passes the Greenwich meridian. Several features stand out.

1. The amplitude of the tide is in most places between 0.25 and 1.5m, *i.e.*, between one-half and three times the amplitude of the equilibrium tide.

2. The phase of the tide does not progress systematically eastward, as we assumed in the above example, except in parts of the Southern Ocean, which is the only part of the world where a disturbance can propagate right around a latitude circle, unobstructed by continents.

3. The greatest amplitudes are along the coasts, especially near gulfs. Correspondingly, there are regions of vanishingly small amplitude (so-called **amphidromic points**) in the middle of the ocean basins. The one exception to these statements is the maximum in the central equatorial Pacific Ocean.

4. The tide progresses systematically around each ocean basin (in fact, around the amphidromic points). For the most part, the progression is clockwise in the southern hemisphere and anticlockwise in the northern hemisphere.

There are two effects that make the tides look so different from our simple channel model. The most obvious is the presence of continents; the second is the Earth's rotation. One effect of the latter (we shall look at other effects later) is to allow waves to become trapped at the coasts.

4.2.4 Kelvin waves

Consider (*cf.* Fig. 4.7) shallow water behavior near a straight coast, in a rotating system.

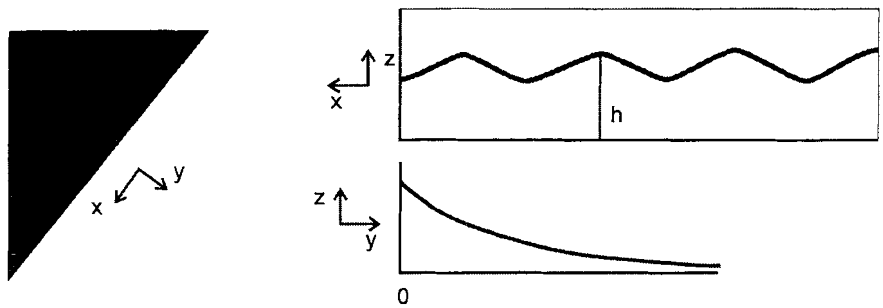

Figure 4.7: Schematic of the trapping of waves a coasts by the planetary rotation. Top figure is in a plane parallel to the coast, which runs along the x-axis; bottom figure normal to the coast.

The eqs. of motion then become (with f the Coriolis parameter)

$$\begin{aligned} \frac{du}{dt} - fv &= -g\frac{\partial h}{\partial x} ; \\ \frac{dv}{dt} + fu &= -g\frac{dh}{dy} ; \\ \frac{dh}{dt} + D\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0 . \end{aligned}$$

Assuming small amplitude perturbations to a basic state with no motion, and uniform depth D gives

$$\begin{aligned} \frac{\partial u'}{\partial t} - fv' &= -g\frac{\partial h'}{\partial x} ; \\ \frac{\partial v'}{\partial t} + fu' &= -g\frac{\partial h'}{\partial y} ; \\ \frac{\partial h'}{\partial t} + D\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) &= 0 . \end{aligned} \tag{4.8}$$

These eqs. have more than one kind of wavelike solution. One such solution—*the Kelvin wave*—is a little strange. The boundary condition at the coast $y = 0$ is $v = 0$: suppose there is a solution with $v = 0$ everywhere. Then

(4.8) become

$$\begin{aligned} \frac{\partial u'}{\partial t} &= -g\frac{\partial h'}{\partial x} \, ; \\ f u' &= -g\frac{\partial h'}{\partial y} \, ; \\ \frac{\partial h'}{\partial t} + D\frac{\partial u'}{\partial x} &= 0 \, . \end{aligned} \tag{4.9}$$

We have left ourselves with 3 eqs in 2 unknowns, which would normally suggest that we are on the wrong track. However, note that the 1st and 3rd of (4.9) are exactly the same two eqs we get in the one-dimensional, nonrotating case. So, just as in the nonrotating case, we get solutions of the form

$$\begin{pmatrix} u' \\ h' \end{pmatrix} = \mathrm{Re}\left\{ \begin{pmatrix} U(y) \\ H(y) \end{pmatrix} \exp\left[ik\,(x - ct)\right] \right\}$$

where $c = \sqrt{gD}$ and where $U = gH/c = cH/D$. However, we have the further constraint of the 2nd of (4.9), which gives

$$\frac{dH}{dy} = -\frac{f}{g}U = -\frac{f}{c}H \, ,$$

which, for constant f, gives[3]

$$H = const \times \exp\left(-\frac{f}{c}y\right) \, . \tag{4.10}$$

The effects of rotation for these Kelvin waves is therefore to *trap* the waves along the coastline, with an e-folding distance of c/f. Otherwise, the motions are entirely parallel to the coast everywhere, and the wave travel at the speed of nonrotating shallow water gravity waves. However, there is one further important implication of (4.10). Nonrotating gravity waves can propagate in either direction. But a physically meaningful solution must *decay* away from the coast (it cannot grow indefinitely as $y \to +\infty$) so we must have $f/c > 0$. In the northern hemisphere ($f > 0$), then, $c > 0$: the wave can only propagate in one direction, with the coast to the right of the direction of propagation (to the left, in the southern hemisphere where $f < 0$).

[3] We will later consider an important case for which f is not constant.

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In the presence of coasts, the tide takes on the characteristics of the kelvin wave. Thus, the tide will tend to propagate anticlockwise around amphidromic points in northern hemisphere ocean basins, and clockwise in the northern hemisphere. Locally, this effect may be counteracted by the tendency of the "open ocean" tide to follow the moon westward, and by interaction between adjacent amphidromic points.

4.2.5 Tides in inlets and bays

Similar behavior is seen on a smaller scale in smaller bodies of water. Figure 4.8 shows the tide in the North Sea. Note the large amplitude, as compared with typical open ocean values, and the similar anticlockwise propagation around the coasts. The propagation is much slower here, consistent with the shallower water.

In such small bodies of water, the effects of gravitational forcing acting directly on the water body are small compared to the indirect effects of open ocean forcing. That is to say, tides in coastal seas and bays are driven primarily by the open ocean tide at the mouth of the bay, rather like driving an organ pipe at a specific frequency by externally playing a note at the end of the pipe. In some cases, this can lead to large amplitudes, by at least two processes. One is simply focusing: if the bay becomes progressively narrower along its length, the tide will be confined to a narrower channel as it propagates, thus concentrating its energy. There are suggestions of this in Fig. 4.8, in the English Channel at the bottom of the figure.

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The second processes is constructive interference between the incoming tide and a component reflected from the coast. Fig 4.9 shows a more spectacular example, the tide in the Gulf of Maine.

The tidal range in the Gulf of Maine is about 3ft at its entrance, but it increases substantially towards the coast and most dramatically in the Bay of Fundy at the NE corner of the Gulf, where the tide exceeds 30ft[4]. We

[4] The mean tide at Burntcoat Head, at the head of the Bay of Fundy, is 38.4ft (11.8m),

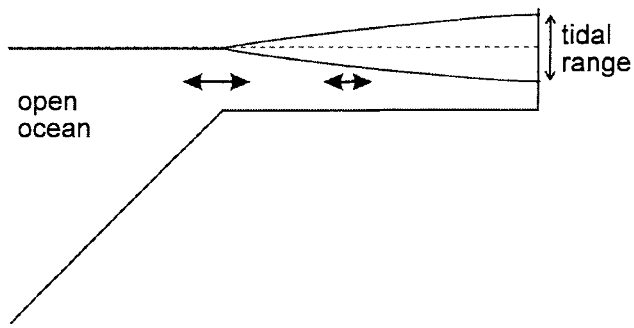

Figure 4.10: Schematic of a quarter-wavelength resonance in a bay.

have seen that a simple reflection can amplify wave amplitude at the coast by a factor of 2, but not 10 or more. What seems to be happening is that the Gulf of Maine/Bay of Fundy system is resonating at the tidal period. This is illustrated in Fig 4.10. Just as in the organ pipe problem, the bay is forced by the tidal currents at its mouth; if the geometry of the bay is such that it takes one-quarter period for a wave to propagate its length, it will support a quarter-wavelength mode at the forcing period, leading to large tides at the head of the bay.

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the highest mean tide in the world.