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12.307 Weather and Climate Laboratory  
Spring 2009

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# Thermal wind

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March 4, 2003

## Abstract

We describe thermal wind balance, derive key equations and discuss the underlying physics.<sup>1</sup>

## 1 The thermal wind equation

Thermal wind is the most fundamental and significant dynamical balance controlling the large-scale circulation of the atmosphere and ocean. It is a consequence of hydrostatic and geostrophic balance, and relates horizontal buoyancy gradients to changes in the horizontal wind with height.

If the flow is such that the Rossby number is small —  $R_o \ll 1$  — where

$$R_o = \frac{U}{fL} \quad (1)$$

(here  $U$  is a typical horizontal current speed,  $f$  is the Coriolis parameter and  $L$  is a typical horizontal scale over which  $U$  varies), then the Coriolis force is balanced by the pressure gradient force in the horizontal component of the momentum equation and the flow is close to geostrophic balance:

$$\mathbf{u}_g = \frac{1}{\rho f} \hat{\mathbf{z}} \times \nabla p, \quad (2)$$

or, writing out its Cartesian components,

$$\begin{aligned} u_g &= -\frac{1}{\rho f} \frac{\partial p}{\partial y}; \\ v_g &= \frac{1}{\rho f} \frac{\partial p}{\partial x}. \end{aligned} \quad (3)$$

If we now combine Eq.(2) with hydrostatic balance, by eliminating the pressure term, we arrive at the thermal wind equation. We now go through this procedure — first for an

incompressible fluid, such as water, and then for a compressible fluid like the atmosphere — to derive various forms of the thermal wind equation.

## 1.1 Thermal wind equation for an incompressible fluid such as water

Suppose that the density of water varies thus:

$$\rho = \rho_o + \delta\rho \text{ and } \frac{\delta\rho}{\rho_o} \ll 1$$

where  $\rho_o$  is a constant reference density, and  $\delta\rho$  is the variation of the density about this reference. Note that this is a very good approximation because, typically, the density of the water in our rotating tank experiments varies about its reference value by less than a fraction of 1%.

Now take  $\partial/\partial z$  of Eq.(3) (replacing  $\rho$  by  $\rho_o$  where it appears in the denominator) we obtain, making use of the hydrostatic relation  $\frac{\partial p}{\partial z} + \rho g = 0$ ,

$$\frac{\partial u_g}{\partial z} = \frac{g}{f\rho_o} \frac{\partial \rho}{\partial y}; \quad \frac{\partial v_g}{\partial z} = -\frac{g}{f\rho_o} \frac{\partial \rho}{\partial x} \quad (4)$$

or, in vector notation

$$\frac{\partial \mathbf{u}_g}{\partial z} = -\frac{g}{f\rho_o} \hat{\mathbf{z}} \times \nabla \rho . \quad (5)$$

So if  $\rho$  varies in the horizontal then the geostrophic current will vary in the vertical. To express things in terms of temperature, and hence derive a connection between the current/wind and the thermal field (called *thermal wind*) we note that the density of water, to an approximation which is useful, depends on temperature  $T$  in a linear fashion:

$$\rho = \rho_o (1 - \alpha (T - T_o)) \quad (6)$$

where  $\alpha$  is the expansion coefficient of water at  $T = T_o$ , and  $T_o$  is a reference temperature. Thus (5) can be written:

$$\frac{\partial \mathbf{u}_g}{\partial z} = \frac{\alpha g}{f} \hat{\mathbf{z}} \times \nabla T \quad (7)$$

This is a simple form of the THERMAL WIND relation. It tells us nothing more than the hydrostatic and geostrophic balance, but it expresses these balances in a different way.

We see that there is an exactly analogous relationship between  $\frac{\partial \mathbf{u}_g}{\partial z}$  and  $T$  as between  $\mathbf{u}_g$

and  $p$ : compare Eqs.(7) and (2). So if we have horizontal gradients of temperature then the geostrophic flow will vary with height.

## 1.2 Thermal wind in pressure coordinates

Eqs.(4) pertain to an incompressible fluid such as the water or the ocean. The thermal wind relation appropriate to the atmosphere can be written down but it is untidy when expressed with height as a vertical coordinate (because of  $\rho$  variations). However it becomes simple when expressed in pressure coordinates. To proceed in  $p$  coordinates, we write the hydrostatic relation thus:

$$\frac{\partial z}{\partial p} = -\frac{1}{g\rho}.$$

and take, for example, the  $p$ -derivative of the  $x$ -component of the geostrophic wind in pressure coordinates —  $(u_g, v_g) = \left(-\frac{g}{f}\frac{\partial z}{\partial y}, \frac{g}{f}\frac{\partial z}{\partial x}\right)$  — yielding:

$$\frac{\partial u}{\partial p} = -\frac{g}{f}\frac{\partial^2 z}{\partial p\partial y} = -\frac{g}{f}\left(\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial p}\right]\right)_p = \frac{1}{f}\frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)_p.$$

Since  $1/\rho = RT/p$ , its derivative *at constant pressure* is

$$\frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)_p = \frac{R}{p}\left(\frac{\partial T}{\partial y}\right)_p,$$

whence

$$\frac{\partial u}{\partial p} = \frac{R}{fp}\left(\frac{\partial T}{\partial y}\right)_p. \quad (8)$$

Similarly, for  $v$  we find

$$\frac{\partial v}{\partial p} = -\frac{R}{fp}\left(\frac{\partial T}{\partial x}\right)_p. \quad (9)$$

Eqs.(8) and (9) express the *thermal wind relationship* in pressure coordinates. In height coordinates, one can obtain a similar relationship, but it is less elegant because of the  $\rho$  factors in Eq.(3).

To see how horizontal gradients of temperature must be accompanied by vertical gradients of wind in the atmosphere, consider Fig.(1). Suppose, for simplicity,  $p = p_0$  is constant at sea level ( $z = 0$ ), and that there is a monotonic decrease of temperature with  $y$  (so  $\partial T/\partial y < 0$ ) within the atmosphere. Then geostrophic balance tells us that  $u = 0$  at  $z = 0$ . Now,

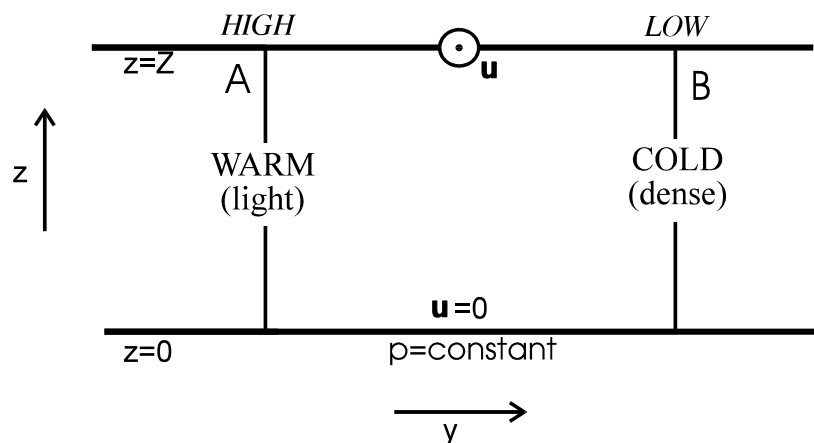


Figure 1: A schematic diagram illustrating why horizontal gradients of temperature must be accompanied by vertical gradients of wind in the atmosphere. Suppose, for simplicity,  $p = p_0$  is constant at sea level ( $z = 0$ ), and that there is a monotonic decrease of temperature with  $y$  (so  $\partial T/\partial y < 0$ ) within the atmosphere. Then geostrophic balance tells us that  $u = 0$  at  $z = 0$ . Since beneath point  $A$  (on the left) the air is warm and therefore light (low density), while is cold and dense beneath point  $B$ , it follows from hydrostatic balance that  $p_A > p_B$ . Aloft, therefore, there must be a geostrophic flow out of the paper and therefore westerly (if we are in the northern hemisphere), with low pressure on its left. Thus, a northward decrease of temperature implies westerly winds increasing with height (i.e.,  $\partial u/\partial p < 0$ ).

hydrostatic balance tells us that, at points  $A$  and  $B$  aloft, the pressure is

$$p(x, y, z) = p_0 - g \int_0^z \rho dz .$$

Since beneath point  $A$  (on the left) the air is warm and therefore light (low density), while is cold and dense beneath point  $B$ , it follows that  $p_A > p_B$ . Aloft, therefore, there must be a geostrophic flow, out of the paper and therefore westerly (if we are in the northern hemisphere), with low pressure on its left. Thus, a northward decrease of temperature implies westerly winds increasing with height (i.e.,  $\partial u/\partial p < 0$ ), consistent with Eq.(8).

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<sup>1</sup>Notes to accompany 12.307: Weather and Climate Laboratory. For a more detailed description see notes on 12.003 web page here: <http://paoc.mit.edu/labweb/notes.htm>