VII. Bedrock Channels: Incision Rates and Longitudinal Profiles

Bedrock Channels are actively incising into rock. Incision rate is set by the ability of flows (and sediment tools carried by the flows) to abrade or "detach" bedrock. In this way they are distinct from transport-limited channels, though in many mixed bedrock-alluvial channels (which are common), this distinction can be blurred.

Transport capacity: Q_c

Sediment Supply (Flux): Q_s

$Q_s / Q_c \rightarrow$ very small

Erosion is governed by ability to "detach" or incision into bedrock, *not* limited by $\frac{\partial q_s}{\partial x}$.

Therefore, erosion is highest where shear stress is highest, rather than where it is increasing most rapidly.

See Whipple, 2004, Annual Reviews in Earth and Planetary Sciences, review paper for background and details.

A. Derivation of a Simple (Generic – not process-specific) Detachment-Limited Incision Model

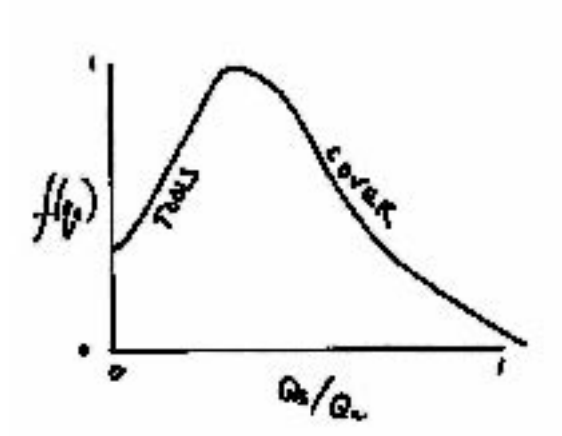
Concept:

Shear Stress model: erosion proportional to shear stress to a power (all below directly analogous in case of unit stream power):

$E = k_b(\tau_b{}^a - \tau_c{}^a)$ or $E = k_b(\tau_b - \tau_c)^a$

$k_b = k_e f(q_s)$; k_e is erosivity at optimum sediment load.

SKETCH of Hypothetical $f(q_s)$ function



Conservation of Mass (water):

$$Q = \bar{u}hW$$

Conservation of Momentum (steady uniform flow):

$$\tau_b = \rho g h S$$

$$\tau_b = \rho C_f \bar{u}^2$$

Goal: write τ_b in terms of slope, discharge.

Use conservation of mass (water), substitute into friction relation:

$$\tau_b = \rho C_f Q^2 (Wh)^{-2}$$

Solve shear stress equation for flow depth, substitute into above gives:

$$\tau_b^{\;3} = \rho^3 g^2 C_f Q^2 (W)^{-2} S^2$$

$$\tau_b = \rho g^{2/3} C_f^{\;1/3} (Q/W)^{2/3} S^{2/3}$$

Can be written as:

$$\tau_b = k_t (Q/W)^{\alpha} S^{\beta}$$

where $k_t = \rho g^{\alpha} C_f^{\;\alpha/2}$; $\alpha = \frac{2}{3}$, $\beta = \frac{2}{3}$ (Generalized Darcy-Weisbach friction relation);

$\alpha = \frac{3}{5}$, $\beta = \frac{7}{10}$ (Manning)

Channel Width Closure

Empirical relation for hydraulic geometry (channel width closure used if direct measurements of W not available):

$W = k_w Q^b$; b ~ 0.5 typical in both alluvial and bedrock rivers.

Substitute into relation for shear stress:

$$\tau_b = k_t k_w^{-\alpha} Q^{\alpha(1-b)} S^{\beta}$$

Drainage Basin Hydrology

For application to ungauged rivers, use empirical relation for basin hydrology:

$$Q = k_q A^c$$

$0.7 \le c \le 1.0$ typical (c < 1 reflects: storm size < basin size, short storm duration, non-uniform ppt, groundwater losses, storage in floodplains, etc)

Combine Above to Derive the Stream Power Incision Model

Substitute into relation for shear stress:

$$\tau_b = k_t k_w^{-\alpha} k_q^{\alpha(1-b)} A^{\alpha c(1-b)} S^{\beta}$$

Combine these for case $\tau_c << \tau_b$ in floods of interest $E = k_e f(q_s) \tau_b^{\,a}$ gives the well-known "Stream Power River Incision Model":

$$E = KA^m S^n$$

$$n = \beta a;\ m = \alpha a c(1-b)$$

$$K = k_e f(q_s) k_w^{-\alpha a} k_q^{\alpha a(1-b)} k_t^{\,a}$$

$\dfrac{m}{n} = \dfrac{\alpha}{\beta} c(1-b)$; $\dfrac{\alpha}{\beta} = 1$ for Generalized Darcy-Weisbach friction relation

m/*n* ~ 0.5 characteristic of broad range of fluvial incision processes that scale with shear stress (or unit stream power) raise to some power (*a*). If erosion process is linear in shear stress (*a* = 1), expected exponents in the stream power incision model are:

m ~ 1/3, n ~ 2/3.

Empirical Field Support

Howard and Kerby, 1983, GSA Bulletin: Empirical study of river incision into weak rocks in badlands over ~20 years.

$\dfrac{dz}{dt} = 0.11 A^{.44} S^{.68}$; $R^2 = .85$ (50 data points)

95% confidence intervals: $0.06 < K < .21$; $.38 < m < .51$; $.58 < n < .78$

Why Stream Power? What is Stream Power? Unit Stream Power?

Both bedrock channel incision and sediment transport have also been proposed to scale with the rate of potential energy expenditure per unit bed area (the so-called “unit stream power”, ω):

$$E = k_b \omega^a$$

Stream power is the rate of potential energy expenditure per unit length of channel. Difference in potential energy between points along a stream is:

$$\Delta P_e = \rho V g \Delta z$$

So rate of change of P_e per unit channel length (Δx) is:

$$\frac{\Delta P_e}{\Delta t \Delta x} = \frac{\rho V g \Delta z}{\Delta t \Delta x} = \rho g Q \frac{\Delta z}{\Delta x} = \rho g Q S = \Omega$$

where we have noted that $Q = V/\Delta t$ and Ω is used for Stream power per unit length.

Stream power per unit bed area (ω) is thus:

$$\omega = \frac{\Omega}{W} = \frac{\rho g Q S}{W}$$

Unit stream power can in fact be directly related to shear stress:

$$\omega = \frac{\rho g Q S}{W} = \frac{\rho g \bar{u} h W S}{W} = \rho g h S \bar{u} = \tau_b \bar{u}$$

Also, recall that average velocity is directly related to bed shear stress:

$$\tau_b = \rho C_f \bar{u}^2 \quad \text{or} \quad \bar{u} = \sqrt{\frac{\tau_b}{\rho C_f}}$$

Thus we have:

$$\omega = \tau_b \bar{u} = \frac{\tau_b^{3/2}}{\sqrt{\rho C_f}} = \frac{\rho g Q S}{W}$$

Therefore, an erosion rule that erosion is a power function of unit stream power can be written, using the hydraulic geometry and basin hydrology from above as:

$$E = k_b \omega^a$$

$$E = K A^m S^n$$

$$n = a\,;\ m = ac(1-b)$$

$$K = k_e f(q_s) k_w^{-a} k_q^{a(1-b)} \rho^a g^a$$

$$\frac{m}{n} = c(1-b)$$

As before, m/n ~ 0.5. If erosion process is linear in unit stream power (a = 1), expected exponents in the stream power incision model are:

m ~ 1/2, n ~ 1.

Thus in conclusion, erosion linear in unit stream power is essentially identical to erosion proportional to shear stress raised to the 3/2 power; in other words the difference is all in the value of the exponent a in the basic postulate: $E = k_e f(q_s)\tau_b^{\,a}$.

B. Conservation of Mass (Rock): Profile Evolution

Now we can consider conservation of mass of the rock to write an evolution equation for a bedrock channel:

$$\frac{\partial z}{\partial t} = U - E = U - KA^m S^n$$

At **steady state**, $U = E$ such that $\frac{\partial z}{\partial t} = 0$, such that we can write:

$$U = KA^m S^n$$

this can be solved directly for the steady-state river slope:

$$S = \left(\frac{U}{K}\right)^{\frac{1}{n}} A^{-\frac{m}{n}}$$

Thus a power-law relation between local channel slope and upstream drainage area is predicted: a straight line in plot of log$S$ vs. log$A$ with slope $-m/n$ (concavity index) and intercept $(U/K)^{1/n}$ (steepness index) (this is true only if U, K, m, and n are spatially uniform … what might happen to profile concavity where $K = K(q_s) = K(x)$?)

Steady-State Longitudinal Channel Profile

By integrating the above relation we can derive an equation for the longitudinal profile of the river at steady state:

$$-\frac{\partial z}{\partial x} = S = \left(\frac{U}{K}\right)^{\frac{1}{n}} A^{-\frac{m}{n}}$$

To integrate we need to write *A* in terms of along-stream distance, *x*. A robust empirical relation known as Hack's law (Hack, 1957) allows this:

$A = k_a x^h$; where ka ~ 6.7 and h ~ 1.67 are typical values.

Substituting in and setting up to integrate both sides:

$$\int \frac{\partial z}{\partial x} = -\int \left(\frac{U}{K}\right)^{\frac{1}{n}} k_a^{-\frac{m}{n}} x^{-\frac{hm}{n}}$$

$$z(x) = -\left(\frac{U}{K}\right)^{\frac{1}{n}} k_a^{-\frac{m}{n}} \left(1-\frac{hm}{n}\right)^{-1} x^{\left(1-\frac{hm}{n}\right)} + C \quad ; \quad \frac{hm}{n} \neq 1$$

$$z(x) = -\left(\frac{U}{K}\right)^{\frac{1}{n}} k_a^{-\frac{m}{n}} \ln(x) + C \quad ; \quad \frac{hm}{n} = 1$$

To find constant of integration, set baselevel *z* = *z*(*L*) at *x* = *L*

$$z(L) = -\left(\frac{U}{K}\right)^{\frac{1}{n}} k_a^{-\frac{m}{n}} \left(1-\frac{hm}{n}\right)^{-1} L^{\left(1-\frac{hm}{n}\right)} + C$$

$$z(x) = \left(\frac{U}{K}\right)^{\frac{1}{n}} k_a^{-\frac{m}{n}} \left(1-\frac{hm}{n}\right)^{-1} \left[L^{\left(1-\frac{hm}{n}\right)} - x^{\left(1-\frac{hm}{n}\right)}\right] + z(L) \quad ; \quad \frac{hm}{n} \neq 1 \quad ; \quad x_c \leq x \leq L$$

$$z(x) = \left(\frac{U}{K}\right)^{\frac{1}{n}} k_a^{-\frac{m}{n}} [\ln(L) - \ln(x)] + z(L) \quad ; \quad \frac{hm}{n} = 1 \quad ; \quad x_c \leq x \leq L$$

where x_c (~ 200-1000m typical) is the distance from the divide at which fluvial processes become dominant over hillslope processes (soil creep, landslides, etc) and debris flow scour.

Fluvial Relief of Drainage Basins

Fluvial Relief is thus given by:

$$R_f = z(x_c) - z(L) = \left(\frac{U}{K}\right)^{\frac{1}{n}} k_a^{-\frac{m}{n}} \left(1-\frac{hm}{n}\right)^{-1} \left[L^{\left(1-\frac{hm}{n}\right)} - x_c^{\left(1-\frac{hm}{n}\right)}\right] \quad ; \quad \frac{hm}{n} \neq 1$$

$$R_f = z(x_c) - z(L) = \left(\frac{U}{K}\right)^{\frac{1}{n}} k_a^{-\frac{m}{n}} [\ln(L) - \ln(x_c)] \quad ; \quad \frac{hm}{n} = 1$$

Note that all except *U*, *K* are geometrical variables, so convenient to simplify to:

$$R_f = z(x_c) - z(L) = \beta \left(\frac{U}{K} \right)^{\frac{1}{n}}$$

where β is expected to vary weakly with tectonic, lithologic, and climatic conditions and is given by:

$$\beta = k_a^{-\frac{m}{n}} \left(1 - \frac{hm}{n} \right)^{-1} \left[L^{\left(1-\frac{hm}{n}\right)} - x_c^{\left(1-\frac{hm}{n}\right)} \right] \quad ; \quad \frac{hm}{n} \neq 1$$

$$\beta = k_a^{-\frac{m}{n}} \left[\ln(L) - \ln(x_c) \right] \quad ; \quad \frac{hm}{n} = 1$$

Channel Profiles and Fluvial Relief – Empirical Geometric Constraints

The above all derived for steady-state conditions where bedrock channel incision is described by the stream power model and *U*, *K*, *m*, and *n* are uniform in space (same tectonics, climate, lithology, and erosion process) – a fairly restrictive set of assumptions. However, it is commonly observed that river profiles follow a power-law relation between channel gradient and upstream drainage area:

$$S = k_s A^{-\theta}$$

where k_s is the steepness index and θ is the concavity index.

Thus the above derivations for profile form and fluvial relief can be repeated for channels with this empirically observed form, yielding equivalent relations that are not directly tied to the above list of assumptions, ie. these relations are valid even if the stream power river incision model is not:

$$z(x) = k_s k_a^{-\theta} (1 - h\theta)^{-1} \left[L^{(1-h\theta)} - x^{(1-h\theta)} \right] + z(L) \quad ; \quad h\theta \neq 1 \quad ; \quad x_c \leq x \leq L$$

$$z(x) = k_s k_a^{-\theta} \left[\ln(L) - \ln(x) \right] + z(L) \quad ; \quad h\theta = 1 \quad ; \quad x_c \leq x \leq L$$

$$R_f = z(x_c) - z(L) = \beta k_s$$

$$\beta = k_a^{-\theta}(1 - h\theta)^{-1}\left[L^{(1-h\theta)} - x_c^{(1-h\theta)}\right] \quad ; \quad h\theta \neq 1$$

$$\beta = k_a^{-\theta}\left[\ln(L) - \ln(x_c)\right] \quad ; \quad h\theta = 1$$

C. River Incision Processes

Powerpoint Presentation: Physical Erosion Processes; plus lecture on constraints on how erosion processes scale with mean bed shear stress (ie. what is exponent *a* for different processes?)

Topics Discussed During Presentation

River Incision into Bedrock:

- Interaction of a suite of process
  - Plucking, Abrasion (bedrock & suspended load), Cavitation (?), Weathering
- Vortices shed off macro-roughness drive processes
  - Relation to mean bed shear stress?
- Critical stress for incision/flood frequency
- Adjustment of channel morphology/bed state
- How non-linear? Relation to Climate?

D. Weaknesses of the Stream Power River Incision Model

- Neglects critical shear stress for incision (assumed exceeded in floods of interest)
- Exponent a and k_b unknown and depend on process(es) active
- k_b should depend on sediment flux – details uncertain
- assumes steady, uniform flow, but much erosion may be related to knickpoints and local flow accelerations – at what scale should S be measured?
- Roughness assumed constant in space (and with flow depth)

- No model for channel width, just assumed to follow hydraulic geometry
- No explicit treatment of flood frequency
- Basin hydrology ($Q \sim A^c$) best for moderate floods. Extreme events can be point-source events
- Small angle approximation breaks down in steep mountain channels and on knickpoints (but minor in comparison to other concerns)

E. Transient Profile Form and Landscape Response Time (time to steady state)

Consider response of landscape starting at an initial steady state and subjected to a sudden step-function change in either rock uplift rate (U) or climate (K). Transient profile consists of two sections separated by an abrupt change in slope – a knickpoint. Downstream of the knickpoint the channel profile is adjusted to the new conditions (steady state with U_f and/or K_f); upstream of the knickpoint the channel profile reflects the old steady-state conditions (steady state with U_i and/or K_i).

The profile reaches steady state when the lower segment reaches $x = x_c$, or when:

$$z(x_c) = z_f(x_c)$$

Time to steady state is given by the ratio of the total change in elevation at $x = x_c$ to the rate of change of elevation at $x = x_c$:

$$T = \frac{distance}{velocity} = \frac{z_f(x_c) - z_i(x_c)}{\partial z(x_c)/\partial t}$$

We have from above that at steady state:

$$z_i(x_c) = \beta\left(\frac{U_i}{K_i}\right)^{\frac{1}{n}} \quad ; \quad z_f(x_c) = \beta\left(\frac{U_f}{K_f}\right)^{\frac{1}{n}}$$

$$\beta = k_a^{-\frac{m}{n}}\left(1 - \frac{hm}{n}\right)^{-1}\left[L^{\left(1-\frac{hm}{n}\right)} - x_c^{\left(1-\frac{hm}{n}\right)}\right] \quad ; \quad \frac{hm}{n} \neq 1$$

$$\beta = k_a^{-\frac{m}{n}}\left[\ln(L) - \ln(x_c)\right] \quad ; \quad \frac{hm}{n} = 1$$

Further, we can deduce from the transient profile form that:

$$\partial z(x_c)/\partial t = const = U_f - E_i(K_f/K_i) = U_f - U_i(K_f/K_i)$$

for a change in U only, $K_f/K_i = 1$; for a change in K only, $U_f = U_i = U$

Thus, defining the fractional change in uplift and the coefficient of erosion as:

$$f_U = U_f/U_i \quad ; \quad f_K = K_f/K_i$$

we may write the rate of change of elevation at $x = x_c$ as:

$$\partial z(x_c)/\partial t = U_i(f_U - 1) \text{ for a change in } U \text{ only}$$

$$\partial z(x_c)/\partial t = U(1 - f_K) \text{ for a change in } K \text{ only}$$

Thus system response time is given simply by:

$$T_U = \frac{z_f(x_c) - z_i(x_c)}{\partial z(x_c)/\partial t} = \frac{\beta K^{\frac{-1}{n}} U_i^{\frac{1}{n}-1}\left(f_U^{\frac{1}{n}} - 1\right)}{(f_U - 1)}$$

$$T_K = \frac{z_f(x_c) - z_i(x_c)}{\partial z(x_c)/\partial t} = \frac{\beta K_i^{\frac{-1}{n}} U^{\frac{1}{n}-1}\left(f_K^{\frac{-1}{n}} - 1\right)}{(1 - f_K)}$$

Assumptions:

- $x_c \neq f(U,K)$; $L >> x_c$ → β = constant
- $S_i = k_{s_i} A^{-\theta}$, $S_f = k_{s_f} A^{-\theta}$ (concavity invariant, k_s function of uplift rate)
    - For stream power model, $k_{s_i} = (\frac{U}{K_i})^{\frac{1}{n}}$, $k_{s_f} = (\frac{U}{K_f})^{\frac{1}{n}}$
- Slope is unchanged above knickpoint

  Retain sharp knickpoint $\Rightarrow$ no information is passed upstream

  T_U 1Ma (order of)

Vertical Knickpoint Velocity

Objective: Use the solution for response time above to solve for vertical knickpoint velocity. Key: knickpoint travels (in z) from the basin outlet to the final position of the fluvial channel head … over the full distance of the new steady-state fluvial relief, in the same total amount of time.

Set this definition of response time equal to the one derived above (they are two ways to express the same thing):

$$T_U = \frac{\Delta z_{\text{knick}}}{V_{kp}} = \frac{z(x_c)_f}{V_{kp}} = \frac{\beta K^{-\frac{1}{n}}(U_f^{\frac{1}{n}} - U_i^{\frac{1}{n}})}{U_f - U_i}$$

$$z(x_c)_f = \beta K^{-\frac{1}{n}} U_f^{\frac{1}{n}}$$

Solve for knickpoint velocity:

$$V_{kp} = \frac{U_f^{\frac{1}{n}}(U_f - U_i)}{U_f^{\frac{1}{n}} - U_i^{\frac{1}{n}}} = \frac{f_U^{\frac{1}{n}} U_i (f_U - 1)}{f_U^{\frac{1}{n}} - 1}$$

where $f_U = U_f / U_i$

thus, $V_{kp} = U_f$ when $n = 1$

- transient, U goes up, K goes down $\Rightarrow$ knickpoint moves upstream at constant vertical rate (all lie on the same contour within a basin!)
- Δ lithology $\Rightarrow$ fixed knickpoint
- Δ uplift (across a fault) $\Rightarrow$ fixed knickpoint

Powerpoint Presentation: Distinctive Transient Behavior of Detachment-Limited and Transport-Limited Models.

F. Advanced Topics: Process-Specific Abrasion Model; Critical Shear Stress and Flood Frequency Distributions

Bedload Abrasion (saltation) Plane Bed (smooth):

Sklar and Dietrich, 2004, WRR

$$E = V_i I_r F_e$$

$$V_i \propto \frac{w_{si}{}^2}{\varepsilon_v}, \quad I_r \propto \frac{Q_s}{WL_s}, \quad F_e = (1 - \frac{Q_s}{Q_c})$$

where Q_s is sediment flux (supply), Q_c is transport capacity and F_e is fraction exposed bedrock.

$$E = \frac{w_{si}{}^2 Q_s}{2\varepsilon_v W L_s}(1 - \frac{Q_s}{Q_c})$$

$$L_s, Q_s, Q_c = f(\tau_b)$$

$Q_s = \beta_g AU$ where β_g is the fraction of sediment that is bedload.

Abrasion by Suspended Load:

$$E_{as} = \frac{S_a q_{ke}}{\rho_r}$$

where ρ_r is rock density, S_a is abrasion susceptibility (ε_v), and q_{ke} is the kinetic energy flux of particles impacting the bed.

$q_{ke} = \frac{1}{2}\rho_r c_v u_p{}^2 \cdot u_p \propto u_p{}^3 \propto u_w{}^3$; where u_p is the particle velocity.

Suspended transport:

$$c_v \propto u_w{}^2,\ q_{ke} \propto u_w{}^5,\ E_a \propto u_w{}^5 \propto \tau_b{}^{5/2},\ a \cong 5/2$$

We can expect: $1 \le a \le \frac{5}{2},\ n = \frac{2}{3}a,\ \frac{2}{3} \le n \le \frac{5}{3},\ \tau_b > \tau_c$

Critical Shear Stress and Flood Frequency Distribution:

Tucker and Bras, 2000, WRR (see more in stochastic_storms_bedrock_chns.ppt)

Snyder et al, 2003, JGR

Tucker, 2004, ESPL

Powerpoint Presentation on the Above Topics.