12.163/12.463 Surface Processes and Landscape Evolution
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VI. Erosional Channel Networks

A. Plan-form Network Properties

Dendritic channel network most common, scale-invariant.

Other forms (trellis, radial patterns, etc.) $\Rightarrow$ lithologic and structural control.

D_d = drainage density

$$D_d = \frac{\Sigma l}{A}[m^{-1}]$$

$$A_c \quad x_c^{\,2}$$

$$\frac{1}{\lambda} \quad \frac{1}{2x_c}$$ λ = spacing between channels

$D_d = f(x_c) \Rightarrow$ lithology, climate, tectonics

Scaling Laws Drainage Networks

Hack's Law (1957):

Length of a channel is related to drainage area.

$l \propto A^{0.52-0.67}$ $\qquad$ $l \propto A^{0.6}$

$A = k_a x^h$ $\quad$ x = along-stream distance $\quad$ $h \quad 1.67$

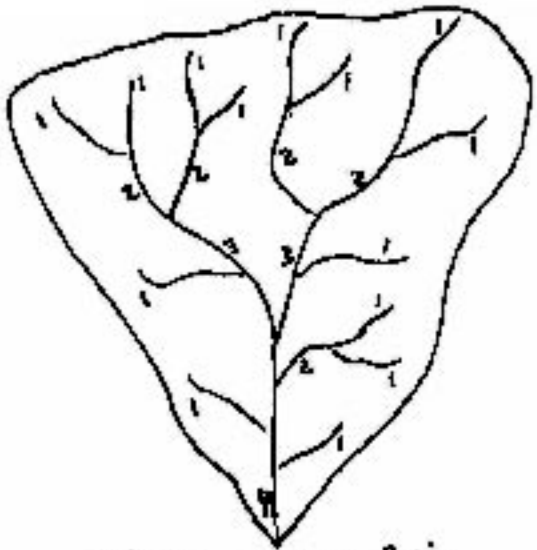
Horton's Laws:

Numbers of channels, lengths, areas all grow in steady geometric progression.

w = order of channel

Second order channel $\rightarrow$ two or more first order channel join

Third order channel $\rightarrow$ two or more second order channel join



4th-Order Drainage Basin
(Dendritic)

Stream numbers (Bifurcation Ratio):

$$R_n = R_b = \frac{n_w}{n_{w+1}} = \text{constant} \quad (3\text{-}5)\ 4 \quad – \quad \text{notation: (range) mode}$$

Stream areas:

$$R_a = \frac{\bar{a}_{w+1}}{\bar{a}_w} = \text{constant} \quad (3\text{-}6)\ 4$$

Stream segment lengths:

$$R_l = \frac{\bar{l}_{w+1}}{\bar{l}_w} = \text{constant} \quad (1.5\text{-}3)\ 2$$

B. Channel Longitudinal Profiles (Empirical):

1960's, 1970's Statistical empirical studies ask what form of equation describes long profile?

Various researchers argue for: Power law $z \propto x^b$, logarithmic, semi-log, exponential forms best-fit channel profiles in their study.

Idea behind this work: downstream increase in $Q_w \Rightarrow$ greater transport capacity/efficiency of erosion

$\Rightarrow$ gentler channel slope

Problem: basin shape governs relation between Q_w and x.

$Q_w = k_q A^c$, $0.7 \le c \le 1$,

c < 1 due to: infiltration losses, non-uniform rainfall, flood plain storage

Combine with Hack's law:

$A = k_a x^h$; $k_a = 6.69$, $h = 1.667$

$Q_w = k_q k_a{}^c x^{hc}$

Thus otherwise identical but different shape basins could have very different river profiles.

Flint (1974) removes problem of basin shape and resolves this debate (more or less)

Flint's law, power-law between *S* and *A*.

$S = k_s A^{-\theta}$

k_s = steepness index, θ = concavity index

Derive estimates from regression of logS vs. logA plots:

$\log S = \log\left(k_s A^{-\theta}\right) = \log k_s + \log A^{-\theta} = \log k_s - \theta \log A$

Commonly observed:

$0.4 \le \theta \le 0.6$; but can vary downstream (spatial variation in geology, tectonics, dis-equilibrium conditions, etc).

Problem: k_s, θ covary … small difference in θ causes huge change in apparent k_s.

Need to normalize somehow for useful intercomparison. Two methods:

$S_r = k_s A_{ref}{}^{-\theta}$; $S = k_{sn} A^{-\theta ref}$

Discussed at length later and in lab.

C. Revisit the Channel Width Problem

Recall empirical observation (robust, all data on both alluvial and bedrock channels):

$$W \propto Q_w^{1/2}$$

Lets consider where this comes from in terms of hydrology and Parker's channel width closure.

Combine conservation of mass of water and Drainage Basin Hydrology and solve for channel width as a function of drainage area and flow velocity and depth:

$$Q_w = \bar{u}hW \;\; ; \;\; Q_w = k_q A^c$$

$$W = \frac{k_q A^c}{\bar{u}h}$$

Write Parker's Channel Width Closure in terms of dimensional shear stress:

$$\tau_* = (1+\varepsilon)\tau_{*c} \;\; ; \; \varepsilon = 0.2 - 0.4 \;\; ; \; \tau_{*c} = 0.06 \text{ \{for gravel\}}$$

$$\tau_* = 2 \text{ \{for sand\}}$$

$$\tau_b = (1+\varepsilon)0.06(\rho_s - \rho)gD \;\; \text{(gravel)}$$

$$\tau_b = 2(\rho_s - \rho)gD \;\; \text{(sand)}$$

$$\tau_b = \alpha(\rho_s - \rho)gD \;\; \text{; where } \alpha \text{ depends on gravel vs. sand bed}$$

Use Conservation of Momentum

$$\tau_b = \rho g h S \;\; ; \;\; \tau_b = \rho C_f \bar{u}^2$$

Use first and second relations for boundary shear stress to substitute in for *h* and velocity, respectively, in the relation for *W*, then substitute in the channel closure condition for boundary shear stress:

$$W = \frac{k_q \rho^{3/2} g \sqrt{C_f} A^c S}{[\alpha(\rho_s - \rho)gD]^{3/2}}$$

Substitute observed relation for river profile concavity:

$$S = k_s A^{-\theta}$$

$$W = \frac{k_q k_s \rho^{3/2} g \sqrt{C_f} A^{c-\theta}}{[\alpha(\rho_s - \rho)gD]^{3/2}}$$

So if $c \sim 1$ (typical) and $\theta \sim ½$, then $W \propto A^{1/2} \propto Q_w^{1/2}$

Empirical relations are internally consistent, which is a nice check, but this does mean that the channel width problem and river profile problem are tightly coupled – solving one requires solving both, ultimately.

D. Transport-Limited Incising Channels (Alluvial but Erosional):

Conservation of Mass (sediment) with Uplift:

$$\frac{\partial z}{\partial t} = U - \frac{1}{1-\lambda_p}\frac{\partial q_s}{\partial x}$$

Generalized Sediment Transport Rule

$$Q_c = Wq_s = K_f A^{m_f} S^{n_f}$$

For gravel bedload, $m_f = 1, n_f = 1$

For sand total load, $m_f = 4/3, n_f = 5/3$

Details of the formulation depend on the channel closure rule, sediment transport equation, treatment of downstream fining, etc {Derivations below}

$\tau_b = (1+\varepsilon)\tau_c$ or $W = k_w Q^b$

Bedload transport: $q_s \propto \tau_b^{3/2}$, $n_f = \frac{2}{3} \times \frac{3}{2} = 1$

Rapid downstream fining: m_f 1.5

Steady-State Profile

By definition for transport-limited channels:

$$Q_c = Q_s$$

Steady-state sediment flux for uniform rock uplift is:

$Q_s = \beta_g AU$; where β_g denotes bedload fraction of total sediment flux

Setting these equal and substituting the general sediment transport rule:

$$K_f A^{m_f} S^{n_f} = \beta_g AU$$

$$S = \left(\frac{\beta_g U}{K_f}\right)^{\frac{1}{n_f}} A^{\frac{1-m_f}{n_f}}$$

IF $m_f = 2, n_f = 2 \Rightarrow \theta \quad \frac{1}{2}$; values given above will be inconsistent.

Suspended sediment transport on hillslopes ($q_s \propto \tau_b{}^3$, no change in width) is consistent with $m_f = 2, n_f = 2 \Rightarrow \theta \quad \frac{1}{2}$ – shown by Willgoose, 1991.

But the relevant question is whether sediment transport relations and channel closure rules for alluvial rivers (not hillslopes) are consistent with observed river profile concavities. To address this, we need to look at how the physics influences the exponents in the generalized transport relation, m_f and n_f.

Gravel Transport (MPM) with Parker Channel Closure

Conservation Mass (Water)

$$Q = \bar{u}hW = k_q A^c$$

Conservation Momentum

$$\tau_b = \rho g h S \quad ; \quad \tau_b = \rho C_f \bar{u}^2$$

combining these gives:

$$\tau_b{}^{3/2} = \frac{\rho^{3/2} g \sqrt{C_f} SQ}{W} = \frac{\rho^{3/2} g k_q \sqrt{C_f} SA^c}{W}$$

Channel closure (Parker, gravel)

$$(\tau_b - \tau_c)^{3/2} = c_w \tau_b{}^{3/2} \quad ; \quad c_w = \left(\frac{\varepsilon}{1+\varepsilon}\right)^{\frac{3}{2}}$$

$$(\tau_b - \tau_c)^{3/2} = \frac{\rho^{3/2} g c_w k_q \sqrt{C_f} SA^c}{W}$$

MPM Gravel Transport (from Alluvial Profile Lectures)

$$Q_c = W q_s = \frac{8W}{\rho^{3/2}((\rho_s - \rho)/\rho)g}(\tau - \tau_{cr})^{3/2}$$

Substitute in relation for $(\tau_b - \tau_c)^{3/2}$:

$$Q_c = \frac{8W}{\rho^{3/2}((\rho_s - \rho)/\rho)g} \frac{\rho^{3/2} g c_w k_q \sqrt{C_f} SA^c}{W}$$

$$Q_c = \frac{8 c_w k_q \sqrt{C_f}}{((\rho_s - \rho)/\rho)} A^c S$$

Thus we find:

$$Q_c = Wq_s = K_f A^{m_f} S^{n_f}$$

$$K_f = \frac{8c_w k_q \sqrt{C_f}}{((\rho_s - \rho)/\rho)} \quad ; \ m_f = c \sim 1 \ ; \ n_f = 1$$

Note: initiation of motion threshold has not been ignored, it has been subsumed into the channel closure rule.

Implication of Steady-State Profile Concavity Index:

$$S = \left(\frac{\beta_g U}{K_f} \right)^{\frac{1}{n_f}} A^{\frac{1-m_f}{n_f}}$$

$$\theta = -\frac{1-m_f}{n_f} = \frac{m_f - 1}{n_f} = 0 \ \text{!! PROBLEM}$$

Sand Transport (Engelund and Hansen) with Parker Channel Closure

Conservation Mass (Water)

$$Q = \bar{u}hW = k_q A^c$$

Conservation Momentum

$$\tau_b = \rho g h S \quad ; \quad \tau_b = \rho C_f \bar{u}^2$$

as before combining these gives:

$$\tau_b^{3/2} = \frac{\rho^{3/2} g k_q \sqrt{C_f}}{W} A^c S$$

Channel closure (Parker, sand)

$$\tau_* = 2 \ \text{\{for sand\}}$$

$$\tau_b = 2(\rho_s - \rho) g D \quad \text{(sand)}$$

Combine above relations to give:

$$\tau_b^{5/2} = \tau_b \tau_b^{3/2} = 2(\rho_s - \rho) g D \frac{\rho^{3/2} g k_q \sqrt{C_f}}{W} A^c S$$

$$\tau_b^{5/2} = \frac{2(\rho_s - \rho)\rho^{3/2} g^2 k_q D \sqrt{C_f}}{W} A^c S$$

Engelund and Hansen Sand Total Load Equation (from Alluvial Profile Lectures)

$$Q_c = Wq_s$$

$$q_{s*} = \frac{0.05}{C_f}\tau_*^{5/2}$$

$$q_{s*} = \frac{q_s}{\sqrt{((\rho_s - \rho)/\rho)gDD}} \quad ; \quad \tau_* = \frac{\tau_b}{(\rho_s - \rho)gD}$$

$$q_s = q_{s*}\sqrt{((\rho_s - \rho)/\rho)gDD} = \frac{0.05\sqrt{((\rho_s - \rho)/\rho)gDD}}{C_f\left[(\rho_s - \rho)gD\right]^{2.5}}\tau_b^{5/2}$$

Substitute the relation for $\tau_b^{5/2}$

$$Q_c = Wq_s = \frac{0.1k_q}{(\rho_s - \rho)\rho\sqrt{C_f}}A^c S$$

Thus we find (again):

$$Q_c = Wq_s = K_f A^{m_f} S^{n_f}$$

$$K_f = \frac{0.1k_q}{(\rho_s - \rho)\rho\sqrt{C_f}} \quad ; \; m_f = c \sim 1 \; ; \; n_f = 1$$

Same prediction (and problem) for steady-state river profile concavity index.

Sand Transport (Engelund and Hansen) with Channel Width ~ A

Conservation Mass (Water)

$$Q = \bar{u}hW = k_q A^c$$

Conservation Momentum

$$\tau_b = \rho g h S \quad ; \quad \tau_b = \rho C_f \bar{u}^2$$

as before combining these gives:

$$\tau_b^{3/2} = \frac{\rho^{3/2} g k_q \sqrt{C_f}}{W}A^c S$$

$$\tau_b^{5/2} = \left(\tau_b^{3/2}\right)^{5/3} = \frac{\rho^{5/2}\left(g k_q \sqrt{C_f}\right)^{5/3}}{W^{5/3}}A^{5c/3}S^{5/3}$$

Engelund and Hansen Sand Total Load Equation (from Alluvial Profile Lectures)

$$Q_c = Wq_s$$

$$q_{s*} = \frac{0.05}{C_f}\tau_*^{5/2}$$

$$q_{s*} = \frac{q_s}{\sqrt{((\rho_s - \rho)/\rho)gDD}} \quad ; \quad \tau_* = \frac{\tau_b}{(\rho_s - \rho)gD}$$

$$q_s = q_{s*}\sqrt{((\rho_s - \rho)/\rho)gDD} = \frac{0.05\sqrt{((\rho_s - \rho)/\rho)gDD}}{C_f\left[(\rho_s - \rho)gD\right]^{2.5}}\tau_b^{5/2}$$

Substitute the relation for $\tau_b^{5/2}$

$$Q_c = Wq_s = \frac{W0.05\sqrt{((\rho_s - \rho)/\rho)gDD}}{C_f\left[(\rho_s - \rho)gD\right]^{2.5}} \frac{\rho^{5/2}\left(gk_q\sqrt{C_f}\right)^{5/3}}{W^{5/3}} A^{5c/3}S^{5/3}$$

$$Q_c = \frac{0.05k_q^{5/3}}{((\rho_s - \rho)/\rho)^2 C_f^{1/6} g^{1/3} DW^{2/3}} A^{5c/3}S^{5/3}$$

Empirical channel closure (Hydraulic Geometry)

$$W = k_w Q^b = k_w k_q^{\,b} A^{bc}$$

Combine above relations to give:

$$Q_c = \frac{0.05k_q^{(5-2b)/3}}{((\rho_s - \rho)/\rho)^2 k_w^{2/3} C_f^{1/6} g^{1/3} D} A^{(5c-2bc)/3}S^{5/3}$$

Thus we find:

$$Q_c = Wq_s = K_f A^{m_f} S^{n_f}$$

$$K_f = \frac{0.05k_q^{(5-2b)/3}}{((\rho_s - \rho)/\rho)^2 k_w^{2/3} C_f^{1/6} g^{1/3} D} \quad ;$$

for $c \sim 1$ and $b \sim ½$; $m_f \sim 4/3$; $n_f = 5/3$

Prediction for steady-state river profile concavity index:

$$\theta = -\frac{1 - m_f}{n_f} = \frac{m_f - 1}{n_f} = \frac{1}{5}$$; Perhaps reasonable for sandy, suspension-dominated alluvial rivers.

D. Transitions from Alluvial to Bedrock (Mixed) Channels

Non-dimensional Bedrock Channel Number , N_{br}

Under what conditions are channels detachment-limited (DL) vs. transport limited (TL)?

Definition: DL: $Q_c > Q_s$, TL: $Q_c \le Q_s$

$$N_{br} = \frac{Q_s}{Q_c}$$

$$N_{br} < 1 \Rightarrow \text{DL (mixed)}, \quad N_{br} \ge 1 \Rightarrow \text{TL}$$

Steady state:

$$Q_s = \beta_g A U,\ Q_c = K_f A^{m_f} S^{n_f}$$

Assume channel is DL, $S = \left(\frac{U}{K}\right)^{\frac{1}{n}} A^{-\frac{m}{n}}$

$$Q_c = K_f A^{m_f} \left[\left(\frac{U}{K}\right)^{\frac{1}{n}} A^{-\frac{m}{n}} \right]^{n_f}$$

Mountain channels $\Rightarrow$ gravel bedload $\Rightarrow n_f = 1$

$$N_{br} = \frac{Q_s}{Q_c} = \frac{\beta_g}{K_f} K^{\frac{n_f}{n}} U^{1-\frac{n_f}{n}} A^{1-m_f+\frac{mn_f}{n}}$$

1. If K goes up (wetter/stormier; weaker rock) or K_f goes down (coarse gravel) or β_g goes up (lots of gravel) $\Rightarrow$ TL
2. If and only if $n < n_f$ $(n < 1)$, does U goes up $\Rightarrow$ DL
3. If $n = n_f$ $(n = 1)$, $N_{br} \ne f(U)$
4. If concavity of TL system is less than concavity of DL $\Rightarrow$ expect transition to TL downstream (vice versa)

   DL: $S = \left(\frac{U}{K}\right)^{\frac{1}{n}} A^{\frac{-m}{n}}$, $\frac{m}{n} \Rightarrow$ concavity (derived in next lectures)

   TL: $S = \left(\frac{\beta_g U}{K_f}\right)^{\frac{1}{n_f}} A^{\frac{1-m_f}{n_f}}$, $-\frac{1-m_f}{n_f} \Rightarrow$ concavity

   Whipple and Tucker (2002) give predictions for the critical drainage area for this transition, and for a given drainage area, the critical rock uplift rate for the transition.
5. If concavities are same $N_{br} \ne f(A)$

Note: in this case, equations for steady-state channel slope above indicate that DL and TL channels can have *identical* longitudinal profiles.

Next Lecture: Idealized Model for Bedrock, Detachment-limited Channel

Transport capacity: Q_c

Sediment Supply (Flux): Q_s

$Q_s / Q_c \rightarrow$ very small

Erosion is governed by ability to "detach" or incision into bedrock.
Not limited by $\frac{\partial q_s}{\partial x}$.