Lecture notes for 12.009, Theoretical Environmental Analysis

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Contents

1 Natural climate change: glacial cycles

1.1 Climatic cycles

Earth's climate has always fluctuated.

Climate fluctuations since the 19th century:

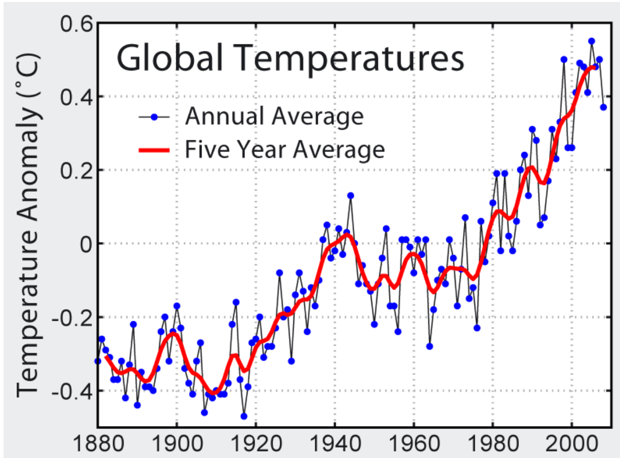
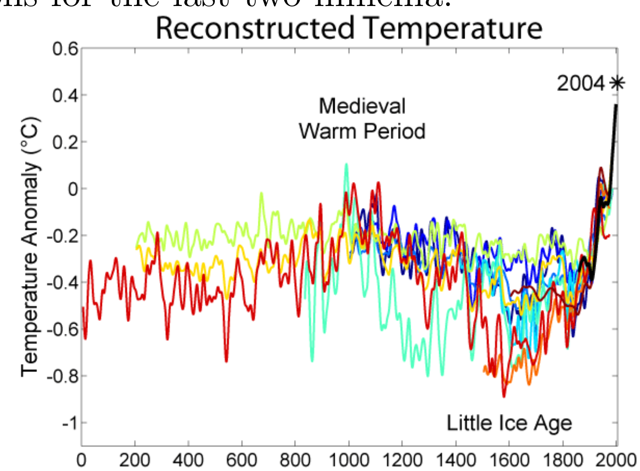


Image created by Robert A. Rohde / Global Warming Art.

Climate fluctuations for the last two millenia:



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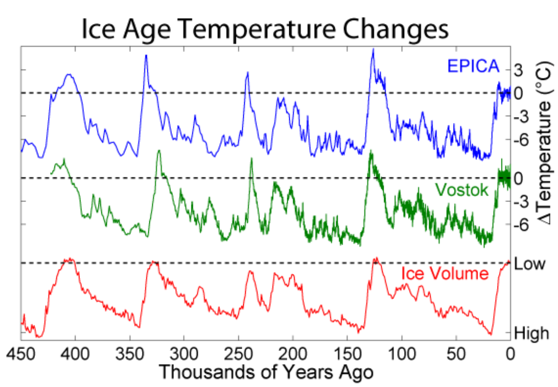
Climate fluctuations for the last 450 Kyr exhibit the 100-Kyr periodicity of *glacial cycles*:



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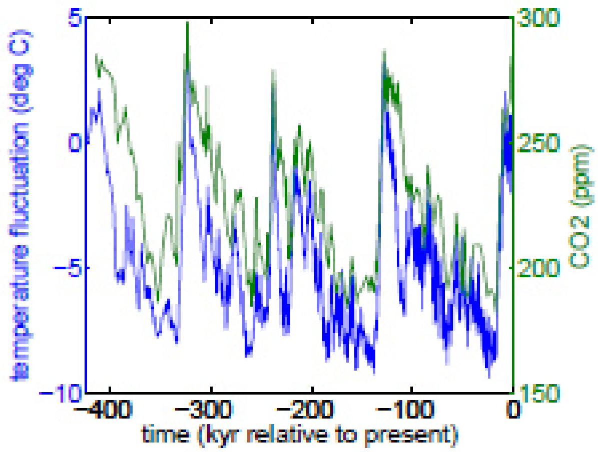
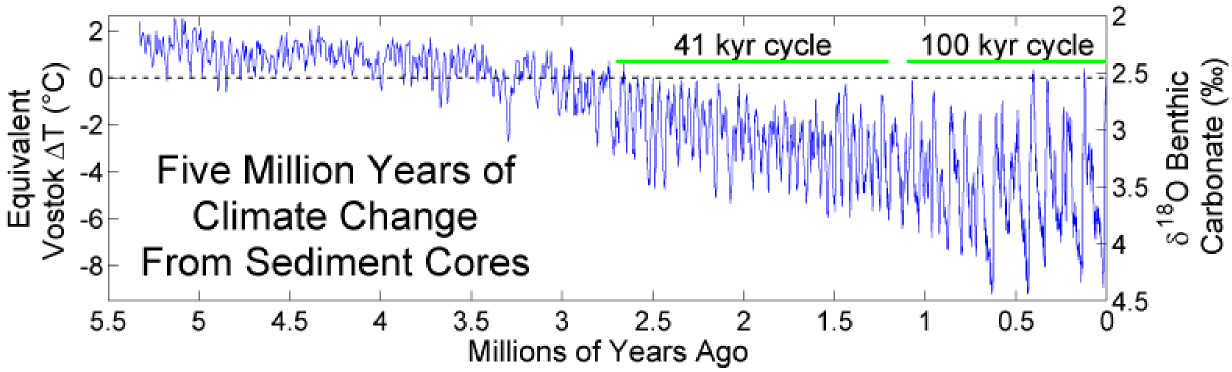
Climate and CO_2 fluctuations for the last 420 Kyr:



This correlation between pCO_2 and climate was highlighted in Al Gore's film *An Inconvenient Truth*. The covariation of these two signals suggests a strong relation between CO_2 and climate, but its explanation remains one of the great unsolved problems of earth science.

Climate fluctuations for the last 5 Myr show that the 100-Kyr cycle began about 1 Ma, and was preceded by the dominance of a 41-Kyr cycle:



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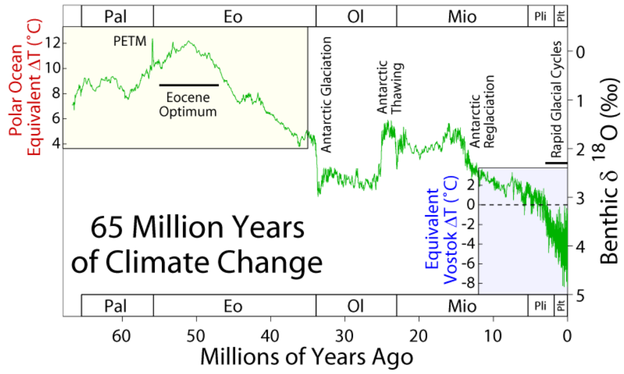
Climate fluctuations for the last 65 Myr:



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Climate fluctuations for the last 540 Myr:

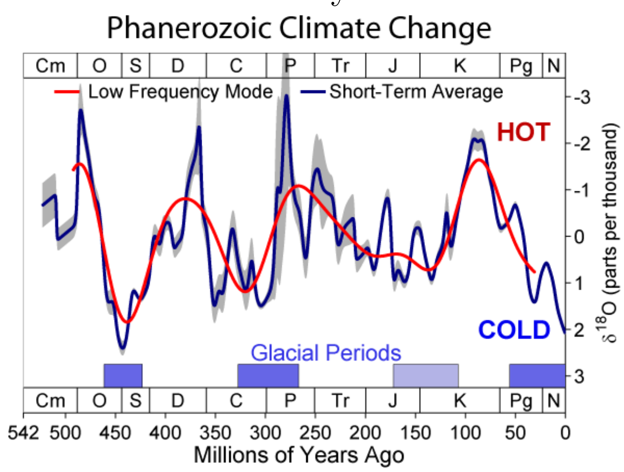


Image created by Robert A. Rohde / Global Warming Art.

1.2 Milankovitch hypothesis: an introduction

Reference: Muller and Macdonald [1].

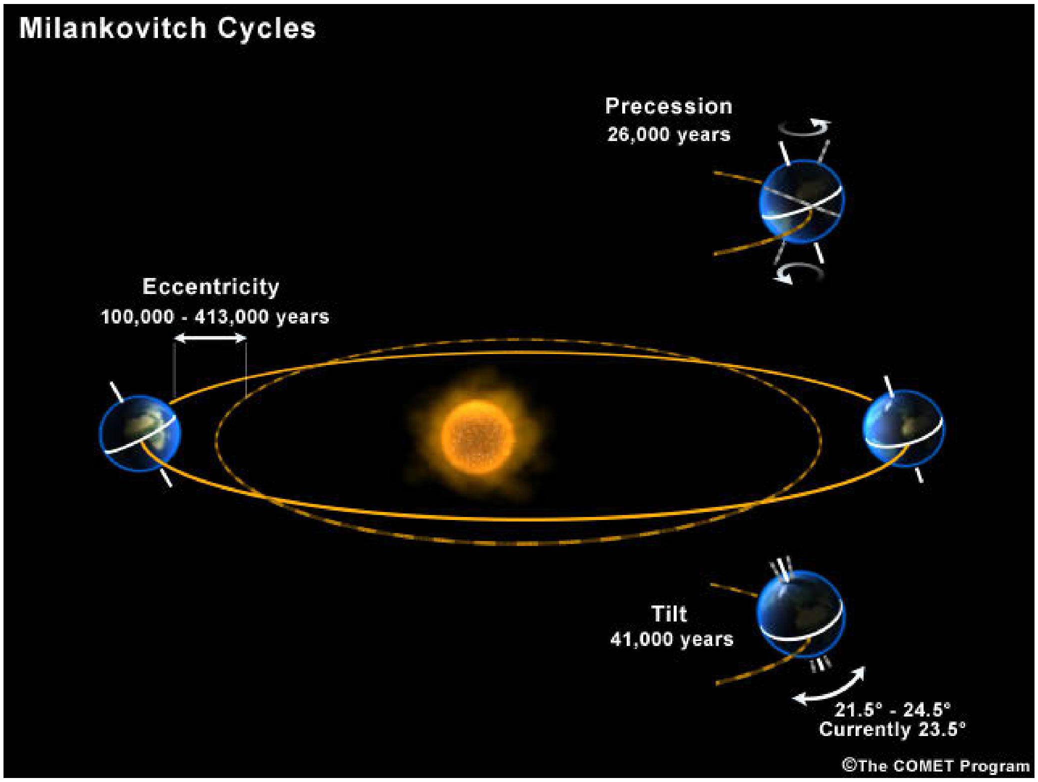
Milutin Milankovitch (1879–1958) proposed that variations in the precession, obliquity, and eccentricity of Earth's orbit are responsible for the glacial cycles.

Similar but less well-developed ideas were proposed in the 19th century by Joseph Adémar and James Croll.

Milankovitch's ideas gained prominence in the 1970s, when evidence of glacial cycles was found in deep sea cores [2].

Let us first take a qualitative look at the three principal orbital parameters.

1.2.1 Precession, obliquity, and eccentricity



www.meted.ucar.edu

Here are movies illustrating precession, obliquity, and eccentricity.

- *Precession* is the slow change in the direction of the North Pole.

  Precession results from torques exerted by the Moon and Sun on Earth's equatorial bulge.

  This movement is analogous to that of a tilted top or gyroscope.

  The period of precession is about 25.8 Kyr.

- *Obliquity* is the angle of the *tilt* of the Earth's pole towards the Sun.

  In other words, it is the angle at which the North Pole tilts towards the Sun in summer.

Today the obliquity is 23.5°. Over the last 800 Kyr it has varied between about 22° and 24.5°.

Obliquity varies with a dominant period of 41 Kyr. Its variations are due to torques from Jupiter (because it is large) and other planets.

This rate of change corresponds to 0.13°/Kyr, which means, e.g., that the Tropic of Cancer—the northernmost latitude at which the Sun may appear directly overhead—has moved 1.4 km in the last 100 yr.

- *Eccentricity* quantifies the deviation of Earth's orbit from a perfect circle. Letting

$$\begin{aligned} A &= \text{major axis of the orbit} \\ B &= \text{minor axis} \end{aligned}$$

The eccentricity ε is

$$\varepsilon = \sqrt{1 - \left(\frac{B}{A}\right)^2}.$$

Today

$$A/B = 1.00014 \qquad \text{and} \qquad \varepsilon = 0.017,$$

i.e., the orbit is within 0.014% of being circular. However the distances of the closest and furthest approaches to Sun are at

$$r_{\text{min}} = \frac{A(1-\varepsilon)}{2} \qquad \text{and} \qquad r_{\text{max}} = \frac{A(1+\varepsilon)}{2}$$

so that

$$\frac{r_{\text{max}} - r_{\text{min}}}{A/2} = 2\varepsilon \simeq 3.3\%.$$

We shall show that eccentricity varies with the angular momentum $L = |\vec{L}|$ of Earth's orbit according to

$$\varepsilon^2 = 1 - kL^2$$

where k is approximately constant. L is maximized when the orbit is circular, and any force that increases L decreases the eccentricity.

The rate of change of angular momentum is related to the torque $\vec{\tau}$ on the Earth-Sun system via

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{\tau}.$$

Torques on the Earth-Sun system arise from any planet that pulls on the two asymmetrically. The major contributions come from Jupiter (because it is large) and Venus (because it is close).

Eccentricity varies between about 0 and 0.05, with periods of 95, 125, and 400 Kyr.

1.2.2 Insolation

The average flux of solar energy at the top of the Earth's atmosphere is

$$S = 1360 \text{ Watts/m}^2.$$

This is the quantity at normal incidence.

But the flux per unit area—the *insolation*— depends on the tilt of a surface with respect to incoming radiation

Taking the Earth's radius to be R_e, we define

$$W = \text{total solar energy flux received by Earth} = \pi R_e^2 S.$$

But this flux is spread out over an area of size $4\pi R_e^2$. Thus the average daily insolation I is

$$I = S/4 = 340 \text{ W/m}^2.$$

Averaged over a year, this quantity varies neither with precession nor obliquity. It does however vary with eccentricity (due to spherical spreading of the radiation).

This does not mean, however, that precession and obliquity are unimportant.

Indeed, Milankovitch proposed that the main driving force of glacial cycles is summer insolation in the northern hemisphere, since two thirds of the Earth's land area is in the north.

The idea is that summer insolation determines the amount of snow melt, and thus the extent of glaciated surface.

The point is that

- Eccentricity determines total insolation.
- Obliquity and precession determine the *distribution* of insolation.

Note also that the effect of precession depends on how close the Earth comes to the sun, which depends on eccentricity.

We introduce the precession angle

$$\omega_M = \text{angle between spring solstice and perihelion.}$$

(Perihelion is the point where the Earth is closest to the sun.)

The effect of precession on insolation is expressed via the *precession parameter*

$$p = \varepsilon \sin \omega_M.$$

The dominant period of variations in p differ from precession itself because of the moving perihelion—the dominant frequencies correspond to periods of 19, 22, and 24 Kyr.

In what follows we provide a series of physical arguments and elementary calculations so that we may better understand variations in insolation and the orbital parameters that make it vary.

1.3 Precession and obliquity

Reference: Kleppner and Kowlankar, pp. 295–301 [3].

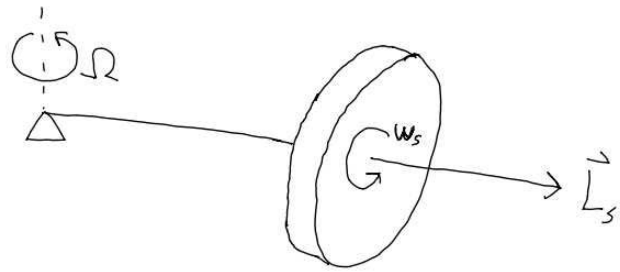
The precession of Earth's axis is analogous to the precession of a gryoscope.

In the following, we show that the uniform precession of a gyroscope is consistent with Newton's laws and the relation between torque and angular momentum (i.e., $\vec{\tau} = \mathrm{d}\vec{L}/\mathrm{d}t$).

We conclude by specifying the analogy with Earth's axial precession.

1.3.1 Gryoscope: horizontal axis

We first suppose that the axis of the gyroscope is horizontal, with one end supported by a free pivot.



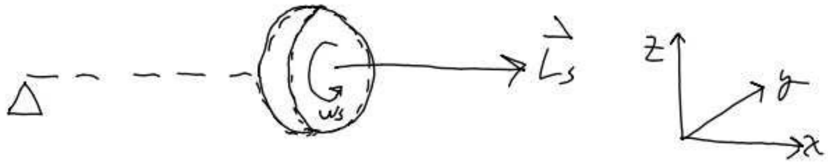
We suppose that the flywheel rotates with angular velocity ω_s.

When the gyroscope is released with a spinning flywheel, it eventually exhibits *uniform precession*, i.e., the axle rotates with constant angular velocity Ω.

Intuitively, we expect that the gyroscope would merely swing vertically about the pivot, like a pendulum. Indeed, this is precisely its behavior when the flywheel does not spin (i.e., $\omega_s = 0$).

But the gyroscope precesses only for large ω_s, i.e., when the flywheel spins rapidly.

In this case virtually all of the gyroscope's angular momentum derives from the spinning flywheel.* Its angular momentum $\vec{L}_s$ is directed along the axle:



The magnitude of $\vec{L}_s$ is

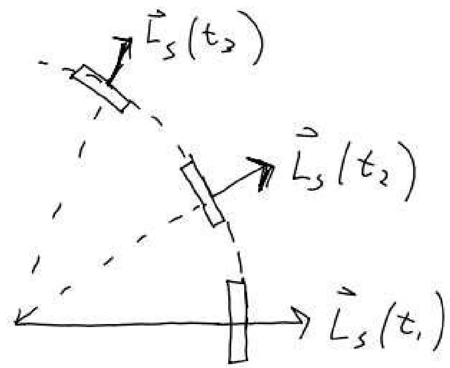
$$|\vec{L}_s| = I_0 \omega_s,$$

where I_0 is the moment of inertia of the flywheel about its axle.†

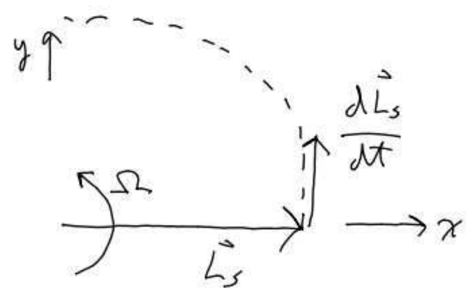
* The small orbital angular momentum is constant for uniform precession.

†Recall the moment of intertia $= \int r^2 \mathrm{d}m$, where r is the distance from the rotation axis and m is mass.

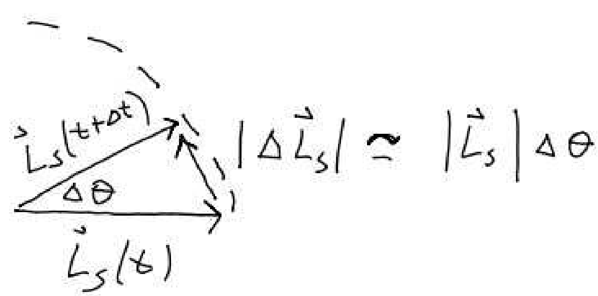
As the gyroscope precesses, $\vec{L}_s$ rotates with it:



Note that $\dfrac{\mathrm{d}\vec{L}_s}{\mathrm{d}t}$ is perpendicular to $\vec{L}_s$:



To determine $\left|\dfrac{\mathrm{d}\vec{L}_s}{\mathrm{d}t}\right|$, we consider small changes in the angular momentum:



Then

$$|\Delta\vec{L}_s| \simeq |\vec{L}_s|\Delta\theta$$

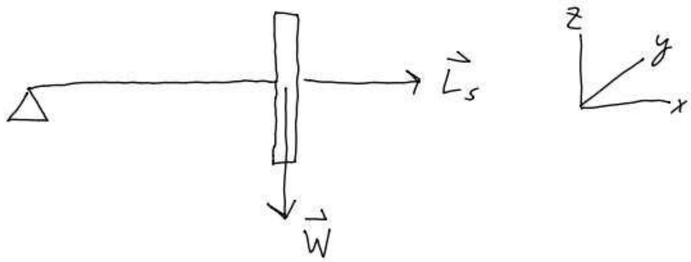
and therefore

$$\left|\frac{\mathrm{d}\vec{L}_s}{\mathrm{d}t}\right| = |\vec{L}_s|\frac{\mathrm{d}\theta}{\mathrm{d}t} = |\vec{L}_s|\Omega.$$

Now recall the relation between the torque $\vec{\tau}$ on a body and its angular momentum $\vec{L}$:

$$\vec{\tau} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}, \quad \text{where} \quad \vec{\tau} = \vec{r} \times \vec{F}.$$

There must therefore be a torque on the gyroscope. We find that it derives from the weight W of the flywheel:



The torque is directed parallel to $\mathrm{d}\vec{L}_s/\mathrm{d}t$, with magnitude

$$|\vec{\tau}| = \ell W,$$

where ℓ is the distance from the pivot to the flywheel.

Since the torque on the gyroscope is

$$\vec{\tau} = \frac{\mathrm{d}\vec{L}_s}{\mathrm{d}t}$$

we have, by substituting on each side our results from above,

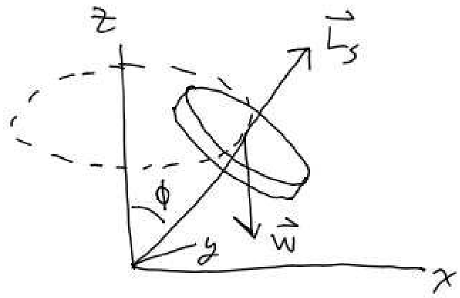
$$\ell W = |\vec{L}_s|\Omega$$

and therefore the angular velocity of precession is

$$\Omega = \frac{\ell W}{|\vec{L}_s|} = \frac{\ell W}{I_0\omega_s}.$$

1.3.2 Gyroscope: tilted axis

Now imagine that the axis of the gyroscope is not horizontal but is instead tilted at an angle ϕ with the vertical:



The vertical (z) component of $\vec{L}_s$ is constant.

The horizontal component varies, but always has magnitude

$$|\vec{L}_s|_{\text{horiz}} = |\vec{L}_s| \sin\phi.$$

Since only the horizontal component contributes to $\mathrm{d}\vec{L}_s/\mathrm{d}t$, we have, reasoning as above,

$$\left|\frac{\mathrm{d}\vec{L}_s}{\mathrm{d}t}\right| = \Omega|\vec{L}_s| \sin\phi.$$

The torque arising from gravity (i.e., $\vec{r} \times \vec{W}$) is again horizontal, but now with magnitude

$$|\vec{\tau}| = \ell W \sin\phi.$$

Using once again that $\vec{\tau} = \mathrm{d}\vec{L}_s/\mathrm{d}t$, we combine the previous two relations to obtain

$$\ell W \sin\phi = \Omega|\vec{L}_s| \sin\phi.$$

We find that the precessional velocity is once again

$$\Omega = \frac{\ell W}{|\vec{L}_s|},$$

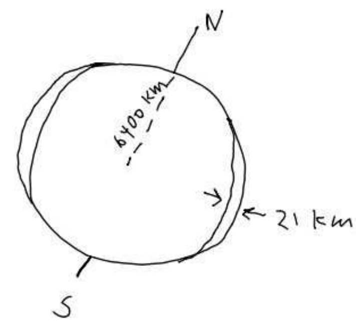
independent of the angle ϕ.

1.3.3 Planetary precession

We now address the precession of Earth's rotation axis.

If the Earth were perfectly spherical and its only interaction were with the Sun, then there would be no torques on it and its angular momentum would always point in the same direction.

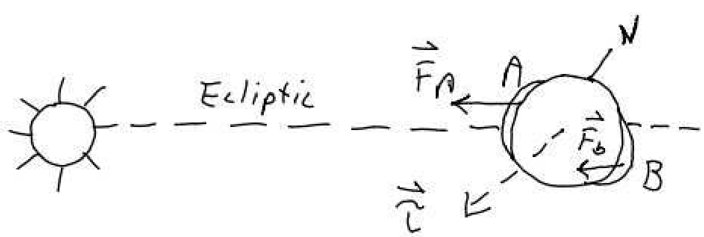
However a torque arises because of the non-spherical shape of the Earth: the mean equatorial radius is about 21 km greater than the polar radius (about 6400 km):



The torque exists because

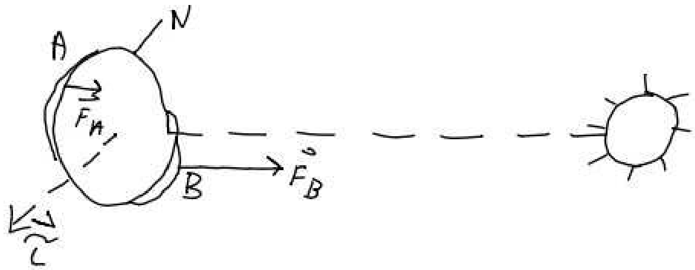
- the Earth's rotation axis is tilted with respect to the orbital plane (the "ecliptic"), by about 23.5°; and
- the Sun pulls asymmetrically on the equatorial bulge.

During the northern hemisphere winter, the bulge above the ecliptic is attracted more strongly to the Sun (F_A) than the bulge below the ecliptic (F_B):



There is thus a counterclockwise torque, out of the plane of the figure.

In summer, B is attracted more strongly to the Sun, but the torque remains in the same direction:



In spring and fall, on the other hand, the torque is zero.

Thus the average torque is in the plane perpendicular to the spin axis, in the plane of the ecliptic.

The moon has the same effect (with about twice the torque).

Consequently the Earth's rotational axis precesses.

The period of the Earth's precession is about 26,000 yr.

Thus, while the Earth's spin axis presently points towards Polaris, this "North Star" will be $2 \times 23.5° = 47°$ off-axis in 13,000 yr.

1.3.4 Obliquity

Whereas precession is the rotation of Earth's spin axis, obliquity is the angle of the axis.

From the preceding discussion, we know that the vertical component of the angular momentum $\vec{L}_s$ due to spin is constant.

However that will only be the case if there are no torques on the Earth outside the Earth-Sun interaction.

We can thus identify changes in Earth's obliquity with torques applied to it.

Aside from the moon, these torques can also come from interactions with other planets, especially Jupiter because it is large, and Venus because it is close, as we discuss at the end of Section 1.4.

Earth's obliquity varies by about $\pm 1^\circ$, with a period of about 41 Kyr.

1.4 Eccentricity

References: Kleppner and Kolenkow, Sect 1.9 and Chap. 9 [3]; Muller and Macdonald [1]

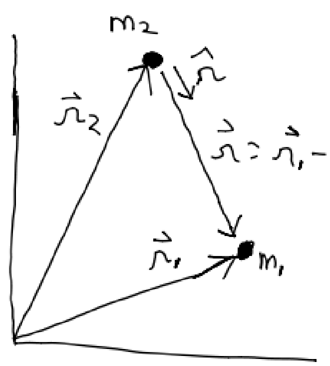
We next analyze the eccentricity of Earth's orbit.

We first examine the problem of *central force motion*, and show that planetary orbits are elliptical.

In doing so, we derive an expression for eccentricity, emphasizing how changes in the Earth's angular momentum can change the eccentricity of its orbit.

1.4.1 Central force motion as a one-body problem

Consider two particles interacting via a force $f(r)$, with masses m_1, m_2 and position vectors $\vec{r}_1$, $\vec{r}_2$.



We define

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \quad (1)$$
$$r = |\vec{r}| = |\vec{r}_1 - \vec{r}_2| \quad (2)$$

For an attractive force $f(r) < 0$, we have the equations of motion

$$\begin{aligned} m_1\ddot{\vec{r}}_1 &= f(r)\hat{r} & (3) \\ m_2\ddot{\vec{r}}_2 &= -f(r)\hat{r}. & (4) \end{aligned}$$

We simplify this system by noting that the center of mass is located at

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}. \qquad (5)$$

Since there are no external forces on the center of mass,

$$\ddot{\vec{R}} = 0$$

and therefore

$$\vec{R}(t) = \vec{R}_0 + \vec{V}t$$

Taking the origin at the center of mass,

$$\vec{R}_0 = 0 \quad \text{and} \quad \vec{V} = 0.$$

We next seek an equation of motion for $\vec{r} = \vec{r}_1 - \vec{r}_2$. We rewrite equations (3) and (4) as

$$\begin{aligned} \ddot{\vec{r}}_1 &= \frac{f(r)\hat{r}}{m_1} \\ \ddot{\vec{r}}_2 &= \frac{-f(r)\hat{r}}{m_2}, \end{aligned}$$

Subtracting the latter from the former, we have

$$\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) f(r)\hat{r}$$

We rewrite this expression as

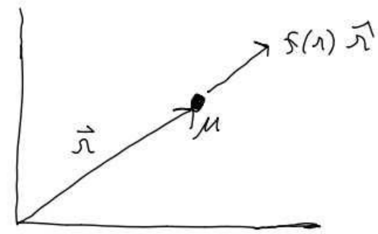
$$\mu\ddot{\vec{r}} = f(r)\hat{r} \qquad (6)$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad (7)$$

is the *reduced mass*.

We have thus reduced the two particle problem to a one-particle problem, described by equation of motion (6) for a particle of mass μ subjected to a force $f(r)\hat{r}$:

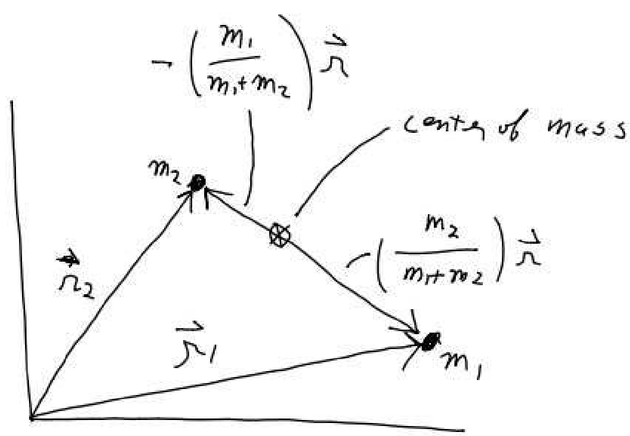


The essential problem is to solve (6) for $\vec{r}(t)$. Then, using (1) and (5), we find the original position vectors

$$\vec{r}_1 = \vec{R} + \left(\frac{m_2}{m_1 + m_2}\right)\vec{r} \quad (8)$$

$$\vec{r}_2 = \vec{R} - \left(\frac{m_1}{m_1 + m_2}\right)\vec{r} \quad (9)$$

where the second term on the RHS of each relation above indicates the position vector relative to the center of mass:



1.4.2 Planar orbits and conserved quantities

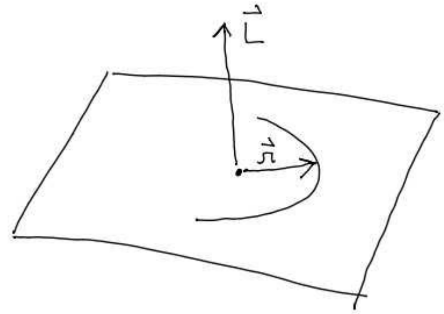
The solution $\vec{r}(t)$ depends on $f(r)$, but some aspects of $\vec{r}(t)$ turn out to be independent of $f(r)$, as we proceed to show.

Planar motion Since $f(r)$ is parallel to $\vec{r}$, it exerts no torque on the reduced mass μ.

Consequently the angular momentum does not change (since $\mathrm{d}\vec{L}/\mathrm{d}t = \vec{\tau} = 0$):

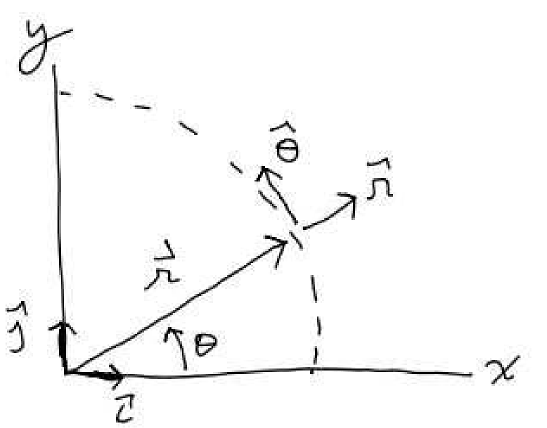
$$\vec{L} = \vec{r} \times \mu\vec{v} = \text{const.}, \qquad \vec{v} = \dot{\vec{r}}.$$

Since the cross product requires that $\vec{r} \perp \vec{L}$, constant $\vec{L}$ requires that $\vec{r}$ must always reside in a plane $\perp \vec{L}$ intersecting the origin.

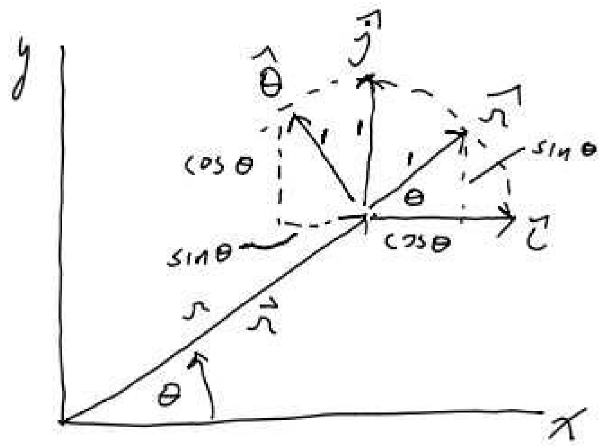


In other words, the motion is confined to a plane, and may therefore be described by just two coordinates.

Representation in polar coordinates We now choose coordinates such that this plane is the xy plane, and introduce polar coordinates r, θ. The associated unit vectors $\hat{r}, \hat{\theta}$ vary with position (unlike the usual Cartesian unit vectors $\hat{\imath}, \hat{\jmath}$:



$\hat{r}$ and $\hat{\theta}$ are straightforwardly related to $\hat{\imath}$ and $\hat{\jmath}$ graphically:



We thus have

$$\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta \quad (10)$$
$$\hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta. \quad (11)$$

We seek an expression for $\ddot{\vec{r}}$ in polar coordinates. Since $\hat{\imath}$ and $\hat{\jmath}$ are fixed unit vectors,

$$\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} = -\hat{\imath}\dot{\theta}\sin\theta + \hat{\jmath}\dot{\theta}\cos\theta \quad (12)$$
$$= \dot{\theta}\hat{\theta} \quad (13)$$

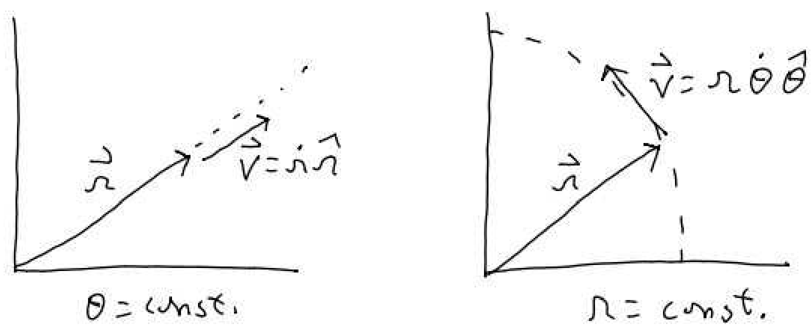
and

$$\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} = -\hat{\imath}\dot{\theta}\cos\theta - \hat{\jmath}\dot{\theta}\sin\theta \quad (14)$$
$$= -\dot{\theta}\hat{r}. \quad (15)$$

The velocity $\dot{\vec{r}}$ is then

$$\dot{\vec{r}} = \frac{\mathrm{d}}{\mathrm{d}t}(r\hat{r}) \quad (16)$$
$$= \dot{r}\hat{r} + r\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} \quad (17)$$
$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}. \quad (18)$$

To see what this means, consider motion in which either θ or r is constant:



When $\theta = \text{const.}$, velocity is radial. Alternatively, when $r = \text{const.}$, velocity is tangential.

We proceed to use these relations to compute the acceleration:

$$\begin{aligned}\ddot{\vec{r}} &= \frac{\mathrm{d}}{\mathrm{d}t}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) \\ &= \ddot{r}\hat{r} + \dot{r}\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t}\end{aligned}$$

Inserting (13) and (15) we obtain

$$\begin{aligned}\ddot{\vec{r}} &= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} &\quad (19) \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} &\quad (20)\end{aligned}$$

The terms proportional to $\ddot{r}$ and $\ddot{\theta}$ represent acceleration in the radial and tangential directions, respectively. The term $-r\dot{\theta}^2\hat{r}$ is the *centripetal acceleration*, and the remaining term, $2\dot{r}\dot{\theta}\hat{\theta}$ is called the *Coriolis acceleration.*

We can now rewrite our one-body equation of motion (6) (i.e., $\mu\ddot{\vec{r}} = f(r)\hat{r}$) in polar coordinates.

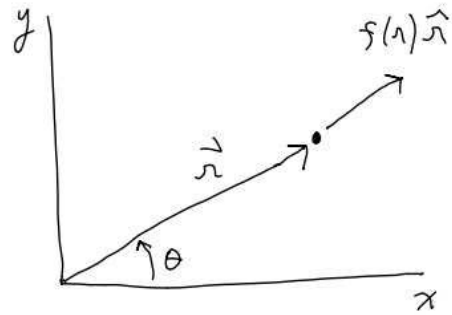
With respect to the radial coordinate $\hat{r}$, we have, after inserting (20), the radial equation of motion

$$\mu(\ddot{r} - r\dot{\theta}^2) = f(r). \quad (21)$$

Likewise, with respect to the angular coordinate $\hat{\theta}$ we have the tangential equation of motion

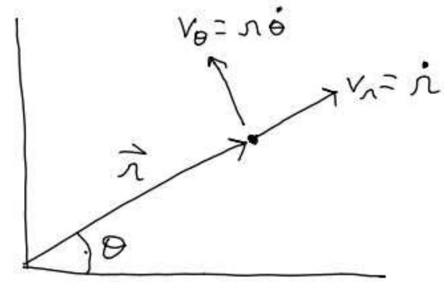
$$\mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0. \quad (22)$$

These relations may look complicated, but they merely describe a particle of mass μ acted upon by a force in the radial direction:



Constants of motion: angular momentum and energy The foregoing development took advantage merely of the constant direction of the angular momentum $\vec{L}$. We now exploit its constant magnitude $l = |\vec{L}|$, and also use the conservation of the total energy E.

We decompose velocity $\vec{v}$ into radial and tangential components:



Since only the angular velocity v_θ contributes to l, we have, using the θ-component of $\dot{\vec{r}}$ from (18),

$$l = \mu r v_\theta = \mu r^2 \dot{\theta}. \tag{23}$$

(Note that computing time derivatives on the LHS and RHS above yields the tangential equation of motion (22).)

The total energy is the sum of the kinetic and potential energies. Using again equation (18), we have

$$\begin{aligned} E &= \frac{1}{2}\mu v^2 + U(r) \\ &= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r). \end{aligned}$$

The potential energy $U(r)$ satisfies

$$U(r) - U(r_a) = -\int_{r_a}^{r} f(r)\mathrm{d}r$$

where $U(r_a)$ is a constant of no physical significance. [Note that, using (22), the radial equation of motion (21) is equivalent to $\mathrm{d}E/\mathrm{d}t = 0$.]

We substitute (23) for $\dot{\theta}$, thereby expressing energy in terms of the angular momentum l:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r).$$

We next define the *effective potential*

$$U_{\text{eff}}(r) = \frac{l^2}{2\mu r^2} + U(r)$$

wherein the first term on the RHS is called the *centrifugal potential.* Then

$$E = \frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r)$$

Rearranging, we have

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{\frac{2}{\mu}(E - U_{\text{eff}}(r))}.$$

We can also obtain $\mathrm{d}\theta/\mathrm{d}t$ directly from the angular momentum (23):

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{l}{\mu r^2}.$$

The orbit of the particle is given by r as a function of θ. We can obtain it by solving

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\dot{\theta}}{\dot{r}} = \frac{l}{\mu r^2}\frac{1}{\sqrt{(2/\mu)(E - U_{\text{eff}}(r))}}. \tag{24}$$

This complete the formal solution of the two-body problem.

1.4.3 Elliptical orbits (Kepler's first law)

In considering Earth's orbit around the sun, note that the mass of the sun is about 330,000 times greater than that of the Earth.

Using the results of Section 1.4.1 and taking m_1 to be the mass of the Earth and m_2 the mass of the Sun, we conclude immediately that the center of mass $\vec{R}$ is essentially at the position of the Sun, which we take to be the origin.

Then, from equations (8) and (9),

$$\begin{aligned} \vec{r}_1 &= \left(\frac{m_2}{m_1 + m_2}\right)\vec{r} \\ &\simeq \vec{r} \end{aligned}$$

and

$$\begin{aligned} \vec{r}_2 &= -\left(\frac{m_1}{m_1 + m_2}\right)\vec{r} \\ &\simeq 0 \end{aligned}$$

since $m_1 \ll m_2$. From equation (7), we also have the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1}{m_1/m_2 + 1} \simeq m_1.$$

In other words, the Earth revolves around the sun as if the sun were fixed at the origin.

For planetary orbits, we have the gravitational interaction

$$U(r) = -G\frac{Mm}{r} \equiv \frac{-C}{r}, \tag{25}$$

where G is that gravitational constant, M the mass of the sun, and m the mass of the planet.

This potential ignores the interactions with other planets. That is, Earth's orbit is not purely the result of a two-body interaction. Indeed, perturbations due to interactions with other bodies are the principal cause of the Milankovitch oscillations—but to understand how these perturbations work, we must first understand the unperturbed problem.

The effective potential is now

$$U_{\text{eff}}(r) = \frac{l^2}{2\mu r^2} - \frac{C}{r},$$

where we retain the use of μ. Inserting into (24) and integrating, we have

$$\begin{aligned}\theta - \theta_0 &= l \int \frac{\mathrm{d}r}{r\sqrt{2\mu E r^2 + 2\mu C r - l^2}} \\ &= \sin^{-1}\left(\frac{\mu C r - l^2}{r\sqrt{\mu^2 C^2 + 2\mu E l^2}}\right),\end{aligned}$$

as may be found, e.g., in a table of integrals. We rewrite the latter expression as

$$\mu C r - l^2 = r\sqrt{\mu^2 C^2 + 2\mu E l^2}\,\sin(\theta - \theta_0).$$

We then solve for r:

$$r = \frac{l^2/\mu C}{1 - \sqrt{1 + 2El^2/\mu C^2}\,\sin(\theta - \theta_0)}.$$

We take $\theta_0 = -\pi/2$ so that $\sin(\theta - \theta_0) = \cos\theta$.

We also define the parameters

$$r_0 = \frac{l^2}{\mu C} \tag{26}$$

and

$$\varepsilon = \sqrt{1 + \frac{2El^2}{\mu C^2}}. \tag{27}$$

When $\varepsilon = 0$, r_0 is the radius of the circular orbit corresponding to l, μ, and C.

The parameter ε is called the *eccentricity* of the orbit. To see why, we rewrite r in terms of r_0 and ε:

$$r = \frac{r_0}{1 - \varepsilon\cos\theta}. \tag{28}$$

We next revert to cartesian coordinates $x = r\cos\theta$, $y = r\sin\theta$. From above, we have that

$$r - \varepsilon r\cos\theta = r_0$$

which is expressed in cartesian coordinates as

$$\sqrt{x^2 + y^2} = r_0 + \varepsilon x.$$

Squaring both sides,

$$x^2 + y^2 = r_0^2 + 2r_0\varepsilon x + \varepsilon^2 x^2$$

and therefore

$$(1 - \varepsilon^2)x^2 - 2r_0\varepsilon x + y^2 = r_0^2.$$

The shape of the orbit depends on ε:

- $\varepsilon > 1$ corresponds to a *hyperbola*. Equation (27) then requires $E > 0$.
- $\varepsilon = 1$ corresponds to a parabola (and $E = 0$).
- $0 \le \varepsilon < 1$ corresponds to an *ellipse*, with

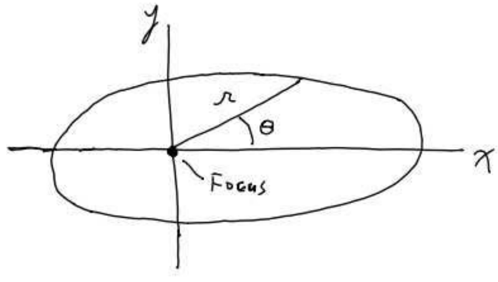
  $$-\frac{\mu C^2}{2l^2} \le E < 0.$$

  The origin is one focus of the ellipse. When $\varepsilon = 0$ the ellipse becomes a *circle*.

The case $0 \le \varepsilon < 1$ corresponds to *Kepler's first law*: planetary orbits are ellipses with the sun at one of the two foci.

The properties of elliptical orbits are of much interest to (Earth's) orbital oscillations.

We return to the polar representation (28).

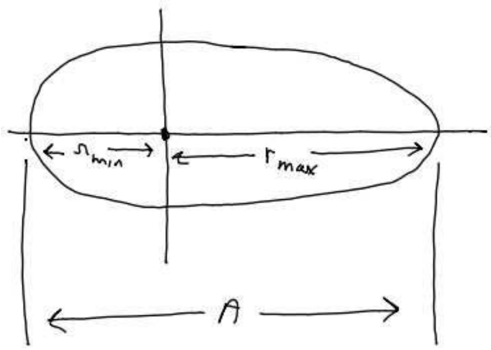


We see that the maximum value of r occurs at $\theta = 0$:

$$r_{\max} = \frac{r_0}{1 - \varepsilon}$$

The minimum value of r occurs at $\theta = \pi$:

$$r_{\min} = \frac{r_0}{1+\varepsilon}$$



The length A of the major axis is therefore

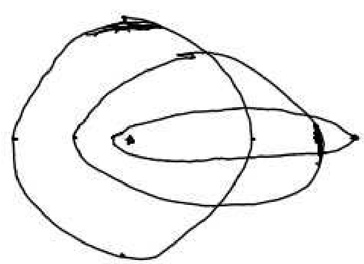
$$\begin{aligned} A &= r_{\min} + r_{\max} \\ &= r_0 \left(\frac{1}{1+\varepsilon} + \frac{1}{1-\varepsilon} \right) \\ &= \frac{2r_0}{1-\varepsilon^2}. \end{aligned}$$

Substituting equations (26) and (27) above, we obtain

$$A = \frac{2l^2/(\mu C)}{1 - [1 + 2El^2/(\mu C^2)]} \tag{29}$$

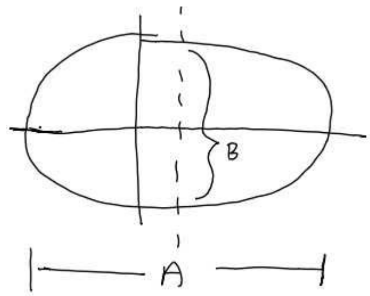
$$= -\frac{C}{E}. \tag{30}$$

Thus the length of the major axis is independent of the angular momentum ℓ and orbits with the same major axis have the same energy E, e.g.:



The minor axis of the ellipse is easily shown to be

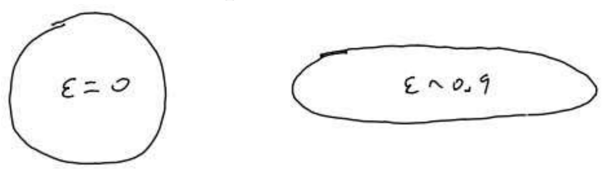
$$B = \frac{2r_0}{\sqrt{1-\varepsilon^2}}. \tag{31}$$



The ratio of the lengths of the major and minor axes is

$$\frac{A}{B} = \frac{2r_0/(1-\varepsilon^2)}{2r_0/\sqrt{1-\varepsilon^2}} = \frac{1}{\sqrt{1-\varepsilon^2}} \tag{32}$$

As ε increases towards 1, the ellipse becomes more elongate:



The present eccentricity of Earth's orbit is small: $\varepsilon = 0.016722$. Thus

$$\left.\frac{A}{B}\right|_{\text{Earth}} = 1.00014,$$

showing that the Earth's orbit is circular within 0.014%..

The difference between the maximum and minimum distances from the sun, however, varies more. Relative to the length of the semi-major axis, we have

$$\frac{r_{\text{max}} - r_{\text{min}}}{A/2} = \frac{2\varepsilon r_0/(1-\varepsilon^2)}{r_0/(1-\varepsilon^2)} = 2\varepsilon,$$

which is about 3.3% for Earth's orbit.

This small difference accounts for changes in solar insolation, as we discuss in Section 1.5.

But first we discuss how the eccentricity or Earth's orbit can change.

1.4.4 Relation of eccentricity to angular momentum

We rewrite the eccentricity equation (27) as

$$\varepsilon^2 = 1 + \frac{2El^2}{m^3M^2G^2}.$$

For elliptical orbits, the energy E is negative.

A classical result in celestial mechanics shows that, when a planet's orbit is perturbed by another body, the major axis A remains invariant to first order in the masses, except for short-period oscillations that do not affect mean behavior.[‡]

Therefore, via equation (30), E can be taken to be effectively constant. Thus

$$k \equiv \frac{-2E}{m^3M^2G^2} \simeq \text{const} > 0,$$

and by rewriting eccentricity as

$$\varepsilon^2 = 1 - kl^2$$

we find that the only way to change ε is to change the magnitude of the angular momentum, l.

Consider the extreme cases:

- $\varepsilon \to 1$. Then $l \to 0$, because the object is falling nearly directly towards the sun, with no transverse velocity.
- $\varepsilon = 0$. Then $l = l_{\max} = k^{-1/2}$ and the orbit is circular.

Thus any force that removes angular momentum makes the orbit more eccentric, and any force that adds it makes the orbit more circular.

[‡]Specifically, A exhibits no *secular* variations that grow like t or $t \sin t$ to first-order in the masses, a result due to Lagrange, following earlier results of Laplace. Poisson later showed that no purely secular variations (growing without oscillating) occur at second order. Periodic oscillations of A do occur at first order, but in the solar system these are all at much shorter periods than concern us here. Further details may be found, e.g., in Section 11.13 of Danby [4] or Chapter 10 of Moulton [5].

The angular momentum $\vec{L}$ changes due to an applied torque $\vec{\tau}$; i.e.,

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{\tau}.$$

Torque on Earth's orbit is produced by planets pulling on the Earth and Sun asymmetrically.

The major torques are those of Jupiter, because it is so large, and Venus, because it is so close.

As a consequence, the eccentricity of Earth's orbit varies between about 0 and 0.05, with periods of 95, 125, and 400 Kyr.

1.5 Insolation

References: Berger [6], Muller and Macdonald [1], Kleppner and Kowlankar [3].

1.5.1 Daily and yearly insolation

The average flux of solar energy at the top of the Earth's atmosphere is

$$S = 1360 \text{ Watts/m}^2.$$

This quantity, called the *solar constant*, is the solar electromagnetic radiation per unit area if it were arriving at normal incidence.

Taking the Earth's radius to be R_e, we define

$$W = \text{total solar energy flux received by Earth} = \pi R_e^2 S.$$

But this flux is spread out over an area of size $4\pi R_e^2$.

Dividing the total flux by the area of the earth, we obtain the average daily *insolation*

$$I = \frac{W}{4\pi R_e^2} = \frac{S}{4} = 340 \text{ W/m}^2.$$

The actual insolation on any given day depends on the distance from the Sun. Let

$$S_a = \text{energy flux received a distance } a \text{ from the sun},$$

where $a = A/2$, the semi-major axis of the elliptical orbit.

When the earth is a distance r from the sun, the average daily insolation is then

$$I(r) = \frac{S_a}{4}\left(\frac{a}{r}\right)^2.$$

where the quadratic factor arises from the spherical spreading of the Sun's radiation.

Over a year of length T, the average insolation is

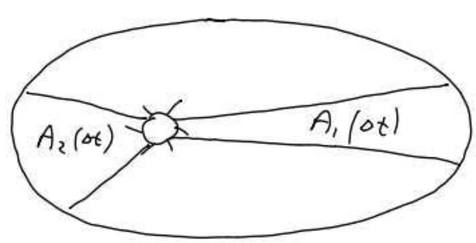
$$I_T = \frac{1}{T}\int_0^T I[r(t)]\mathrm{d}t = \frac{S_a}{4T}\int_0^T \left(\frac{a}{r}\right)^2 \mathrm{d}t. \tag{33}$$

To calculate this integral, we must first derive *Kepler's second law.*

1.5.2 Kepler's second law

(We have already derived Kepler's first law in Section 1.4.3: planetary orbits are elliptical, a conseqence of the $1/r^2$ gravitational force.)

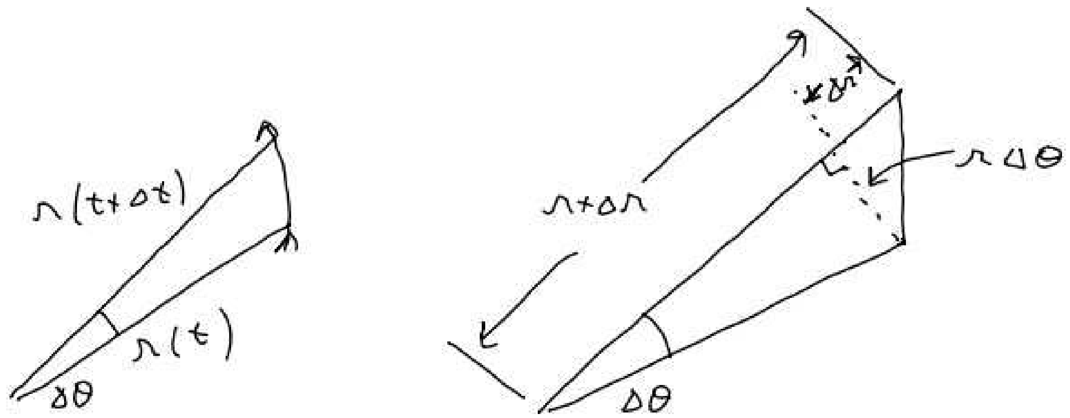
Kepler's second law states that the area $\mathcal{A}$ swept out by the radius vector from the sun to a planet in a given length of time is constant throughout the orbit:



In other words, $\mathcal{A}_1 = \mathcal{A}_2$ and, more generally, $\dfrac{\mathrm{d}\mathcal{A}}{\mathrm{d}t} = \text{const.}$.

To show this, we note that a small change in area, $\Delta\mathcal{A}$, due to small change Δr and $\Delta\theta$ is

$$\begin{aligned}\Delta\mathcal{A} &\simeq \frac{1}{2}(r+\Delta r)(r\Delta\theta)\\ &= \frac{1}{2}r^2\Delta\theta + \frac{1}{2}r\Delta r\Delta\theta\end{aligned}$$



Then

$$\begin{aligned}\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t} &= \lim_{\Delta t\to 0}\frac{\Delta\mathcal{A}}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{1}{2}\left(r^2\frac{\Delta\theta}{\Delta t} + r\frac{\Delta r\Delta\theta}{\Delta t}\right)\\ &= \frac{1}{2}r^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\end{aligned}$$

where we have neglected the small second order term representing the tiny triangle.

Now note that the angular momentum of the Earth relative to the sun is

$$\vec{L} = \vec{r}\times m\vec{v}$$

From equation (18), the velocity

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}.$$

Consequently

$$\begin{aligned}\vec{L} &= \vec{r}\times m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})\\ &= mr^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{k}\end{aligned}$$

since $\hat{r} \times \hat{\theta} = \hat{k}$. Substituting the expression above into that for $\mathrm{d}\mathcal{A}/\mathrm{d}t$, we have

$$\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t} = \frac{l}{2m} = \text{const.} \tag{34}$$

Recalling that the angular momentum l is a constant for the orbit (Section 1.4.2), we thus arrive at Kepler's second law.

1.5.3 Relation of insolation to eccentricity

We return now to the computation of the annually averaged insolation I_T, and thus the integral $\int (a/r)^2 \mathrm{d}t$ of (33). From the results we have just obtained, we have

$$\begin{aligned}\frac{r^2}{2}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{\text{area of ellipse}}{T} \\ &= \frac{\pi ab}{T},\end{aligned}$$

where $b = B/2$, the semi-minor axis, and T is the duration of a year.

From equation (32), we have $b = a\sqrt{1-\varepsilon^2}$; therefore

$$\frac{r^2}{2}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\pi a^2\sqrt{1-\varepsilon^2}}{T}.$$

We rewrite this expression as

$$\left(\frac{a^2}{r^2}\right)\mathrm{d}t = \frac{T}{2\pi\sqrt{1-\varepsilon^2}}\,\mathrm{d}\theta$$

Substituting this result into equation (33), the annually averaged insolation, we obtain

$$I_T = \frac{S_a}{4T}\int_0^{2\pi} \frac{T}{2\pi\sqrt{1-\varepsilon^2}}\,\mathrm{d}\theta$$

Since for Earth's orbit, ε varies only from about 0.0 to 0.05 in 100 Kyr, to good approximation it is constant over one year (T). Thus

$$I_T = \frac{S_a}{4\sqrt{1-\varepsilon^2}}. \tag{35}$$

We previously observed, in Section 1.4.4, that the major axis A is effectively constant. Consequently S_a can be taken constant.

The annually averaged insolation I_T therefore depends only on the eccentricity.

Since ε is small, we can expand I_T to second order about $\varepsilon = 0$:

$$\begin{aligned} I_T(\varepsilon) &= \frac{S_a}{4}\left(1 + \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{1}{\sqrt{1-\varepsilon^2}}\bigg|_{\varepsilon=0} \cdot \varepsilon + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\frac{1}{\sqrt{1-\varepsilon^2}}\bigg|_{\varepsilon=0} \cdot \varepsilon^2 + \dots\right) \\ &= \frac{S_a}{4}\left(1 + \frac{\varepsilon^2}{2} + \mathcal{O}(\varepsilon^4)\right). \end{aligned}$$

Thus increasing eccentricity from 0 to 0.05 produces an increase in the relative yearly insolation by a factor of about $0.05^2/2$, or about 0.1%.

This small change can be understood from the figure below equation (30): as eccentricity increases, about half the orbit becomes further away from the Sun, while the other half is closer. Thus the changes almost cancel.

We can get a sense of what the actual changes mean by recalling, from the beginning of this section, that the average daily insolation is 340 $\mathrm{W/m^2}$.

Thus the increase in daily insolation due to increasing eccentricity is much less than 1 $\mathrm{W/m^2}$.

In contrast, the effective change in radiative forcing due to other changes is much larger:

effect	equivalent radiative force ($\mathrm{W/m^2}$)
average daily insolation	340
average reflected insolation (albedo)	−53.5
clouds	−28
doubling CO_2	4

Consequently changing eccentricity has only a minor impact on radiative forcing.

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[2] Hayes, J. D., Imbrie, J. & Shackleton, N. J. Variations in the earth's orbit: Pacemaker of the ice ages. *Science* **194**, 1121–1132 (1976).

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[5] Moulton, F. R. *An Introduction to Celestial Mechanics, Second revised edition* (Dover Publications, New York, 1970).

[6] Berger, A. & Loutre, M. F. Precession, eccentricity, obliquity, insolation, and paleoclimates. In Duplessy, J.-C. & Spyridakis, M.-T. (eds.) *Long-Term Climatic Variations*, vol. 122 of *NATO ASI Series*, 107–151 (Springer-Verlag, Heidelberg, 1994).

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