12.005 Lecture Notes 12

Displacement Gradients

Quantitative description:

Suppose

$$D \rightarrow D'$$

$$P(x_i) \rightarrow P'(x_i + u_i)$$

$$Q(x_i + dx_i) \rightarrow Q'(x_i + dx_i + u_i + du_i)$$

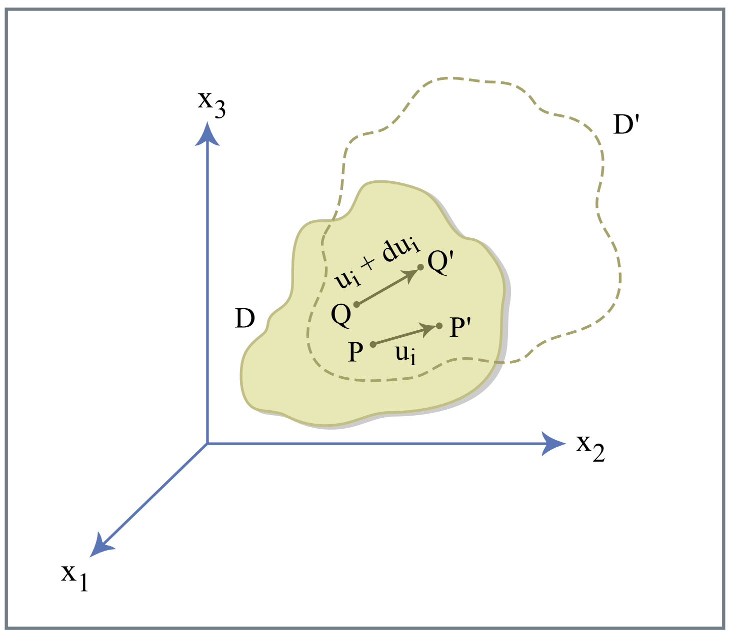

Figure 12.1

Figure by MIT OCW.

Suppose:

The deformation is continuous.

The first derivative $\frac{\partial u_i}{\partial x_j}$ are continuous and very small.

Then chain rule $\Rightarrow$

$$du_i = \frac{\partial u_i}{\partial x_1} dx_1 + \frac{\partial u_i}{\partial x_2} dx_2 + \frac{\partial u_i}{\partial x_3} dx_3 = \frac{\partial u_i}{\partial x_j} dx_j$$

Note: $\dfrac{\partial u_i}{\partial x_j}$ relates two vectors du_i and dx_j and is therefore a second rank tensor.

$$\begin{bmatrix} du_1 \\ du_2 \\ du_3 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2} & \dfrac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

For the Ventura Basin results shown, for the 2-D solution, for the stations HOPP-HAPY-SNP,

$$\frac{\partial u_i}{\partial x_j} = \begin{bmatrix} 0.2 & 0.45 \\ 0.08 & -0.48 \end{bmatrix} \times 10^{-6} \text{ each year}$$

Note: We have no sensitivity to rigid body translations $\left(\dfrac{\partial u_i}{\partial x_j} \equiv 0 \right)$.

What about rotations?

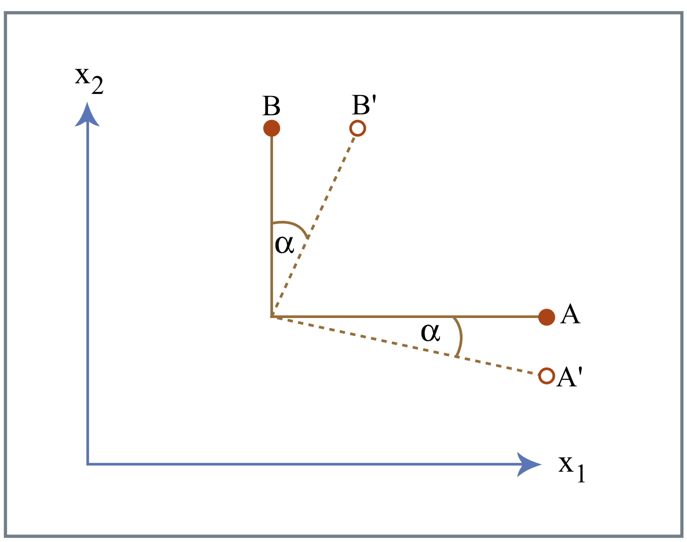

Figure 12.2

Figure by MIT OCW.

$$\frac{\partial u_2}{\partial x_1} = -\tan\alpha \ \rightarrow \ -\alpha$$

$$\frac{\partial u_1}{\partial x_2} = \alpha$$

$$\alpha = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) \equiv \omega_{12}$$

So part of this displacement gradient tensor just gives rigid body rotation.

Rewriting

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

where

$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = \varepsilon_{ij}$ This is strain. It is symmetric.

$\Rightarrow$ six independent components

$\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = \omega_{ij}$ This is rotation. It is antisymmetric.

$\Rightarrow$ three independent components

For Ventura Basin, each year

$$\varepsilon_{ij} = \begin{bmatrix} 0.20 & 0.26 \\ 0.26 & -0.48 \end{bmatrix} \times 10^{-6}$$

$$\omega_{ij} = \begin{bmatrix} 0 & 0.19 \\ -0.19 & 0 \end{bmatrix}$$

Interpretation of ε_{ij}

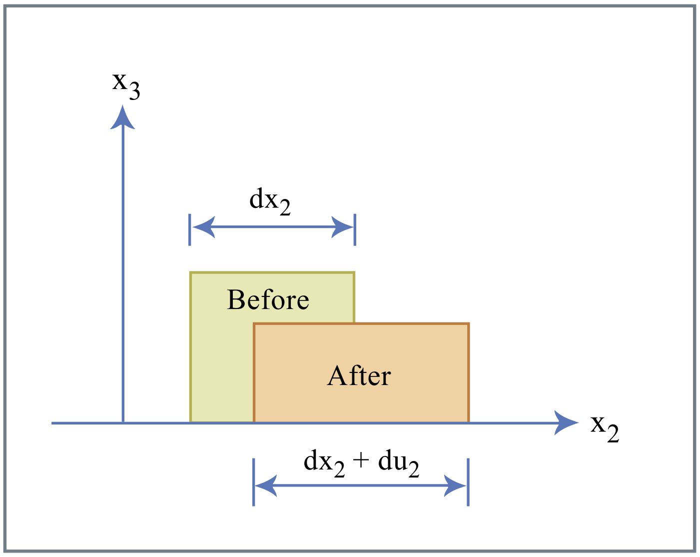

Figure 12.3

Figure by MIT OCW.

For example,

$$\varepsilon_{22} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right) = \frac{\partial u_2}{\partial x_2}$$

$$dx_2 + du_2 = \left(1 + \frac{\partial u_2}{\partial x_2}\right) dx_2 = \left(1 + \varepsilon_{22}\right) dx_2$$

Ventura Basin shows

N-S direction shortening

E-W direction lengthening

Change in shape

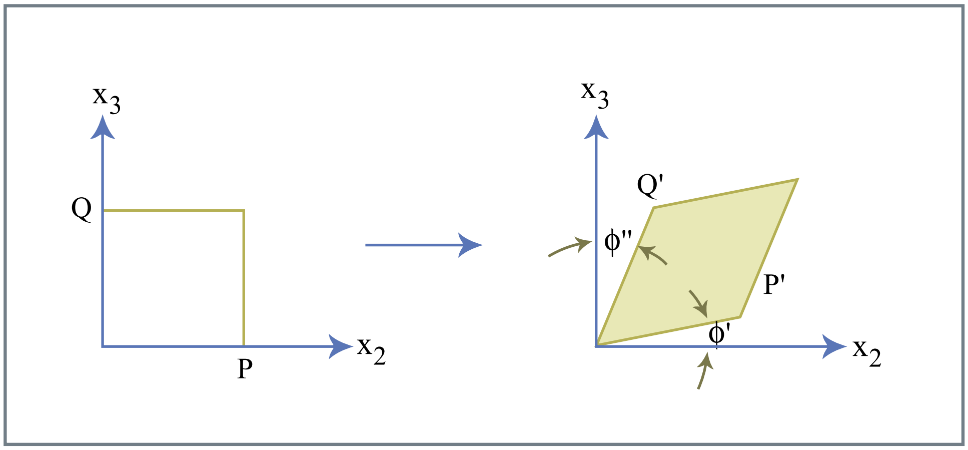

Figure 12.4

Figure by MIT OCW.

$$\phi' \simeq \tan\phi' = \frac{\partial u_3}{\partial x_2}$$

$$\phi'' \simeq \tan\phi'' = \frac{\partial u_2}{\partial x_3}$$

$$\phi' + \phi'' = \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} = 2\varepsilon_{23}$$

where ε_{23} is ½ distortion of x_2, x_3 axes.

$2\varepsilon_{23} \equiv \gamma_{23}$ It is called “engineering” strain.