Lecture Notes on Fluid Dynmics
(1.63J/2.21J)
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3-6unsteadyBL.tex

3.6 Unsteady boundary layers

Let us begin from the full momentum equation

$$\vec{q}_t + \vec{q}\cdot\nabla\vec{q} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{q} \tag{3.6.1}$$

Let the veloicty and times scales be U_o and T, the tangential length scale be L and the transverse length scale be $\delta \sim \sqrt{\nu T}$. Hence the suitable normalization is

$$\begin{aligned} x' &= x/L, \quad y' = y/\sqrt{\nu T}, \quad t' = t/T, \\ u' &= u/U_o, \quad v' = \frac{vL}{U_o\delta} = \frac{v}{U_o}\sqrt{\frac{L^2}{\nu T}}, \\ p &= \frac{pT}{\rho U_o L}, \quad U' = U/U_o. \end{aligned} \tag{3.6.2}$$

If primes are omitted for brevity, the dimensionless equations are,

$$u_x + v_y = 0, \tag{3.6.3}$$

$$u_t + \frac{U_oT}{L}(uu_x + vu_y) = -p_x + \frac{\nu T}{L^2}u_{xx} + u_{yy} \tag{3.6.4}$$

$$\frac{\nu T}{L^2}\left[v_t + \frac{U_oT}{L}(uv_x + vv_y)\right] = -p_y + \frac{\nu T}{L^2}\left[\frac{\nu T}{L^2}v_{xx} + v_{yy}\right] \tag{3.6.5}$$

Outside the viscous boundary layer,

$$U_t + (\frac{U_oT}{L})UU_x = -\frac{1}{\rho}p_x \tag{3.6.6}$$

Two parameters control the motion: U_oT/L (inertia) and $\nu T/L^2$ (viscosity).

Several scenarios are possible:

1. Low amplitude and slow motion: $U_oT/L \ll 1, \nu T/L^2 = O(1)$. The tangential and transverse scales are comparable. To the leading order, the approximate equations in physical coordinates are

$$u_x + v_y = 0, \tag{3.6.7}$$

$$\vec{q}_t = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{q} \tag{3.6.8}$$

This is just the Oseen's approximation.

2. Finite amplitude, fast motion, $U_o T/L = O(1), \nu T/L^2 \ll O(1)$. The boundary layer is thin. To the leading order, nonlinearity is important in the boundary layer.

$$u_x + v_y = 0, \tag{3.6.9}$$

$$u_t + \frac{U_o T}{L}(uu_x + vu_y) = -p_x + u_{yy} = U_t + \frac{U_o T}{L} U U_x + u_{yy} \tag{3.6.10}$$

or, in physical coordinates,

$$u_t + (uu_x + vu_y) = U_t + UU_x + \nu u_{yy} \tag{3.6.11}$$

3. Small-amplitude and fast motion. $\nu T/L^2 \ll U_o T/L \ll 1$. This is a limit of the preceding case; linearization is possible. Examples are : the initial stage of transient motion starting from rest, oscillating flow around a vibrating body, or wave motion (sound or sea waves) past a body (or a droplet, a bubble), etc.