Quiz1 Solutions

1. Problem 1

Part (a)

Let A = Area of *inner region of coverage.*

$$E[A] = E[|X_1 - X_2| \cdot |Y_1 - Y_2|] = E[|X_1 - X_2|] \cdot E[|Y_1 - Y_2|] = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

Part(b)

Let variance of $A = \sigma^2 = E[A^2] - E[A]^2$. Therefore,

$$E[(|X_1 - X_2| \cdot |Y_1 - Y_2|)^2] - (\frac{1}{9})^2 = E[|X_1 - X_2|^2] \cdot E[|Y_1 - Y_2|^2] - (\frac{1}{9})^2$$

Here

$$E[|X_1 - X_2|^2] = E[|Y_1 - Y_2|^2] = \int_0^1 x^2 2(1-x)dx = \frac{2}{3} - \frac{2}{4} = \frac{1}{6}$$

Thus

$$\sigma^2 = (\frac{1}{6})^2 - (\frac{1}{9})^2 = \frac{5}{324}$$

Part(c)

$$F_A(a) = Pr(A \le a) = Pr(D_x D_y \le a)$$

$$F_A(a) = 1 - \int_a^1 \int_{a/x}^1 4(1-y)(1-x)dydx$$

Problem 2

Part(a)

Answer =

$$\frac{\lambda}{\lambda + \mu_1}$$

This is the probability that a new customer will arrive before service to the first customer in the

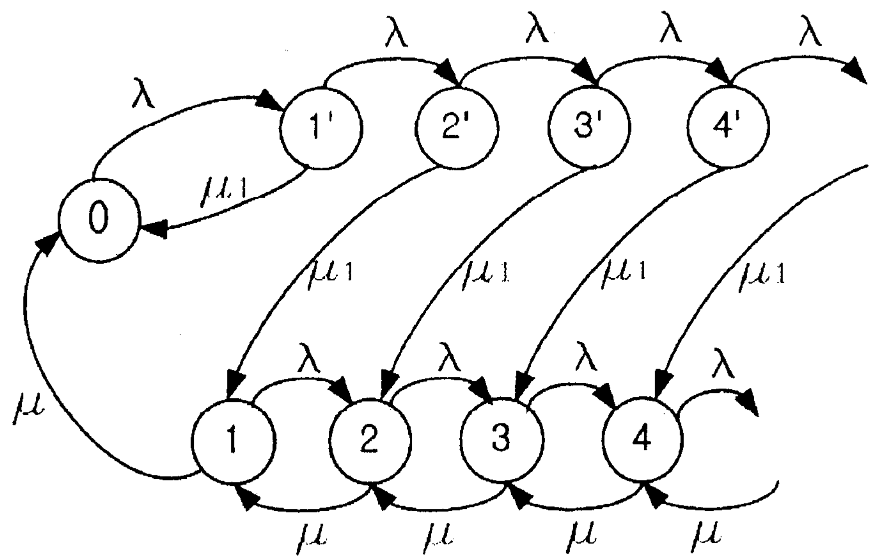

Figure 1: Problem 2

busy period is completed.

Part (b) There are two ways this can happen.

- **Prob1** After first customer arrives: new customer arrives - new customer arrives - three service completions.
- **Prob2** After first customer arrives: new customer arrives -¿ service completion - new customer arrives - two service completions.

$$Prob(1) = (\frac{\lambda}{\mu_1 + \lambda})^2 (\frac{\mu_1}{\mu_1 + \lambda})(\frac{\mu}{\mu + \lambda})^2$$

$$Prob(2) = (\frac{\lambda}{\mu_1 + \lambda})^2 (\frac{\mu_1}{\mu_1 + \lambda})^2 (\frac{\lambda}{\mu + \lambda})(\frac{\mu}{\mu + \lambda})^2$$

Therefore, answer is

$$Prob(1) + Prob(2)$$

Part (C) Answer is 'Yes.' This is sufficient. The condition $\lambda < \mu$ prevents the queue from becoming infinitely long. Even if $\lambda >> \mu_1$, all that will happen is that the probability of having busy periods with more than one customer will be high.

Part (d) Let a state with a prime (1',2',3'...) indicates a state in which the first customer after an idle period is still receiving service. States 1,2,3... are states in which the first customer after an idle period has already left the system.

Problem 3

Part (a) Suppose we have cdf $F_X(x)$ and pdf $f_x(x)$ and we are told that $Y = aX$, where $a > 0$. Here $Y = 200X$, i.e., $a = 200$. That is, we have scaled all distances from 1 to 200.

$$F_Y(y) = Pr(Y < y) = Pr(aX < y) = Pr(X < y/a) = F_X(y/a) = 1/2 + \frac{1}{\pi}tan^{-1}(y/a)$$

Thus,

$$f_Y(y) = \frac{d}{dy}F_X(y/a) = \frac{1}{a}f_X(y/a)$$

So, before truncation, we have

$$f_Y(y) = \frac{1}{200\pi}\frac{1}{1+(y/200)^2}$$

for all values of y. Truncation event $= T = -200 < Y < 200$.

$$Pr(T) = F_Y(200) - F_Y(-200) = \frac{1}{2} + \frac{1}{\pi}tan^{-1}(200/200) - (\frac{1}{2} + \frac{1}{\pi}tan^{-1}(-200/200))$$

$$Pr(T) = \frac{1}{\pi}\frac{\pi}{4} - (-\frac{1}{\pi}\frac{\pi}{4}) = \frac{1}{2}$$

So, we can finally write,

$$f_{Y|T}(y|T) = f_Y(y)/Pr(T) = (1/100\pi)\frac{1}{1+(y/200)^2} \qquad \text{for } -200 < y < 200$$

Part (b) Mean = 0 by symmetry and finiteness of the pdf. Variance is finite by bounded pdf.

Part (c) Question asks for

$$Pr(70 < y < 140|T) = \frac{F_Y(140) - F_Y(70)}{Pr(T)} = \frac{2}{\pi}\left(tan^{-1}(140/200) - tan^{-1}(70/200)\right)$$