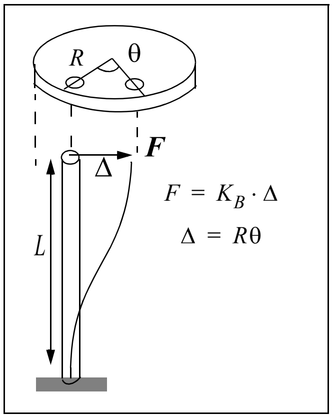
Structures Design Task - Engineering Analysis - Final Revision
1.101 Sophomore Design - Fall 2006

Bending

Each of the *n* shafts will be subject to bending as the top plate rotates relative to the fixed, bottom



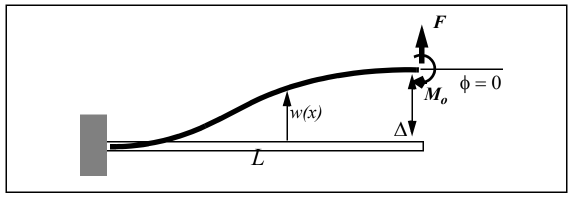


plate. For the shaft considered as a beam, we have when the slope at $x = L$ is zero. We have

$$F = K_B \cdot \Delta \qquad \text{where the stiffness} \qquad K_B = \frac{12 \cdot EI}{L^3}$$

Recall that this was obtained from an engineering beam theory analysis of the deflection of the beam. Insisting that the slope of the beam is zero at $x = L$ gave us the following relationship between the “applied” force F and the end moment M_o, namely

$$M_{o'} = F(L/2)$$

We use this later in figuring the bending stresses acting at the left end of the beam.

Δ, the transverse displacement at the end, can be expressed in terms of the angle of rotation of the top plate as $\Delta = R\theta$ so

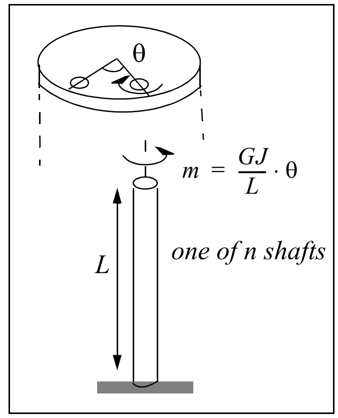
$$F = K_B \cdot R\theta$$

The torque required to rotate the top plate about its central axis for the case when there are *n* identical shafts, under bending (but no torsion) is then:

$$T = nK_B \cdot R^2\theta$$

Torsion

We have $m = (GJ/L) \cdot \theta$ where m is the torque applied to each shaft. *(GJ/L)* is the "stiffness" of each shaft.



This is our major result. Observe

- We can express the shear stress and strain distribution in terms of the applied moment

$$\tau(r) = r \cdot (m/J)$$
$$\gamma(r) = r \cdot m/(GJ)$$

- Our analysis is identical for a hollow shaft. All of the symmetry arguments apply. Only the expression for *J* changes: It becomes

$$J = \int_{Area} r^2 dA = (\pi/2)(R_o^4 - R_i^4)$$

  where R_0 is the outer radius and R_i the inner radius of the shaft.

- If we do anything to destroy the rotational symmetry, all bets are off. In particular if we slit a hollow tube lengthwise we dramatically decrease the torsional stiffness of the tube.

For the design task, with "n" shafts, we have

$$T = nK_T \cdot \theta \qquad \text{where} \qquad K_T = (GJ)/L$$

For combined bending and torsion, we have, superimposing the two resistances to rotation:

$$T = n \cdot (K_B R^2 + K_T) \cdot \theta$$
$$\text{where}$$
$$K_B = \frac{12 \cdot EI}{L^3} \qquad \text{and} \qquad K_T = \frac{GJ}{L}$$

In these: $J = 2I = \pi r^4/2$ - for solid shaf*t* - *r* is the radius of the shaft, not to be confused with *R*.

and $G = \frac{E}{2 \cdot (1 + \upsilon)}$ is the "shear modulus" and ν is Poisson's ratio which you can take as 1/3.

Structure Stiffness

In class, we let $K_T = \lambda \cdot K_B R^2$ where if λ =1, torsion and bending of the shafts contribute equally to the overall stiffness. Expressing the K's in terms of the physical properties and geometry, we obtained $\boxed{\frac{L}{R} = 4 \cdot \sqrt{\lambda}}$

With this, we figured the torque T in terms of θ to be for $\lambda = 1$:

$$T = \frac{3}{32} \cdot n\pi E\frac{r^4}{R} \cdot \theta \qquad \text{or, for abitrary lambda} \qquad T = \frac{3}{64} \cdot n\left(\frac{1+\lambda}{\lambda\sqrt{\lambda}}\right)\pi E\frac{r^4}{R} \cdot \theta$$

We put this stiffness in terms of L, rather than R, if you like, using the above, and obtain

$$T = \frac{3}{8} \cdot n\pi E\frac{r^4}{L} \cdot \theta \qquad \text{or, for abitrary lambda} \qquad T = \frac{3}{16} \cdot n\left(\frac{1+\lambda}{\lambda}\right)\pi E\frac{r^4}{L} \cdot \theta$$

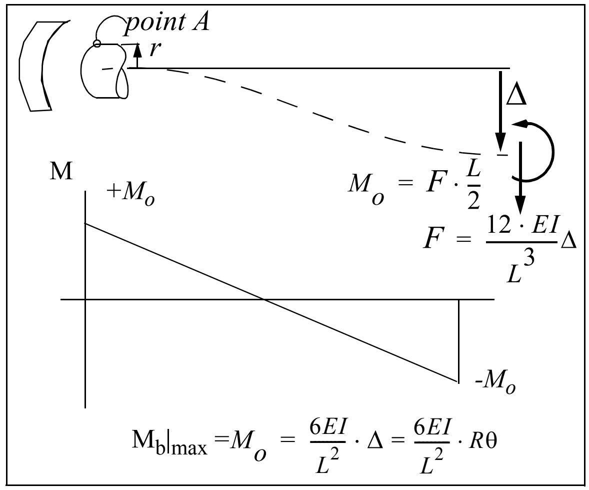
Finally, if the contribution of torsion of the shafts is zero (free to rotate relative to the top plate), we have[1].

$$T = \frac{3}{16\lambda} \cdot n\pi E\frac{r^4}{L} \cdot \theta \qquad = \frac{3}{4} \cdot \left(\frac{n}{\lambda}\right)\frac{EI}{L} \cdot \theta$$

1. Note: The prior edition of this handout had an error in this relationship; the lambda was missing (it only applied for λ =1). Note in this "corrected" edition, the second form was still in error. Again it was missing a lambda in the denominator. So *this*, hopefully final, revision has corrected *that* mistake.

Stress considerations

Due to Bending



The normal stress due to bending varies linearly over the cross section, reaching a maximum value at a distance *r* from the axis of the shaft. It is related to the bending moment by

$$\sigma = \frac{Mr}{I}$$

The bending moment distribution is shown in the figure. The maximum bending moment occurs at the ends of the shaft

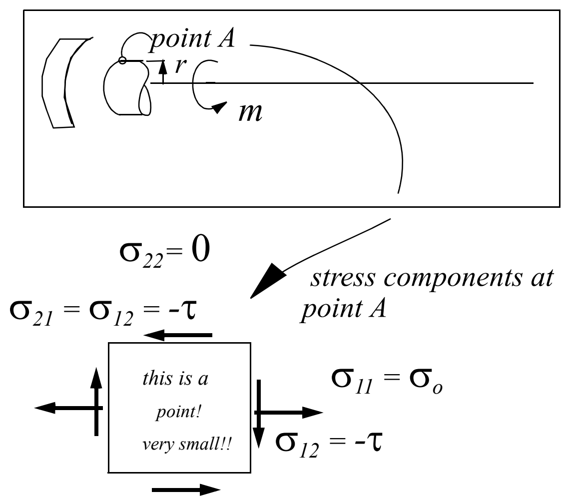
$$M_o = \frac{6EI}{L^2} \cdot \Delta = \frac{6EI}{L^2} \cdot R\theta$$

The maximum (tensile) stress due to bending occurs at *point A*[1]

$$\sigma = 6E\left(\frac{Rr}{L^2}\right) \cdot \theta = \frac{3E}{8\lambda}\left(\frac{r}{R}\right) \cdot \theta$$

(or at the bottom of the beam, at the other end).

Due to Torsion



If the shafts are fixed at their ends to the top plate, then the shaft rotates about its axis the same amount as the whole structure (assuming small displacements and rotations so the torsion doesn't "couple" with the bending). The shear stress distribution does not vary along the shaft but does vary (linearly) over the cross section, reaching a maximum at the outer radius, e.g., at *point A*. From prior notes on torsion:

$$\tau = G\left(\frac{r}{L}\right) \cdot \theta = \frac{E}{2 \cdot (1 + \upsilon)}\left(\frac{r}{L}\right) \cdot \theta$$

The figure at the right shows 3 of the 6 components of the (symmetric) stress tensor at the *point A* where both the shear and the bending stress components, τ and σ, are maximum. The 3 components acting on the surface in the plane of this

1. The second form of this expression for sigma was in error. The factor (r/L) is now (r/R), the rightness of which you can verify by eliminating L^2 from the first form.

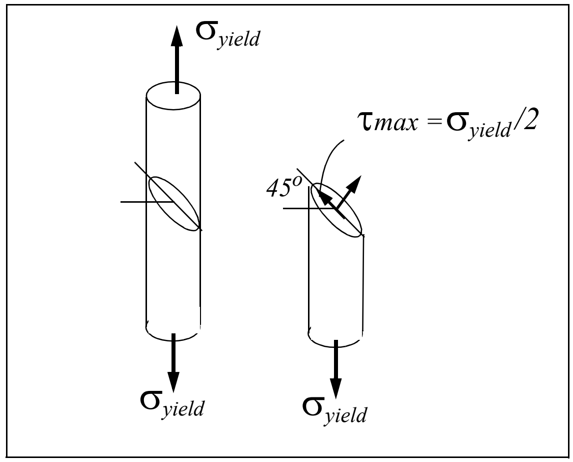
paper must all vanish because the surface of the shaft is “stress free”. (This means also that the “transverse shear stress due to bending” which is represented by the component σ_{13}, must also vanish at point A).

Mohr's Circle

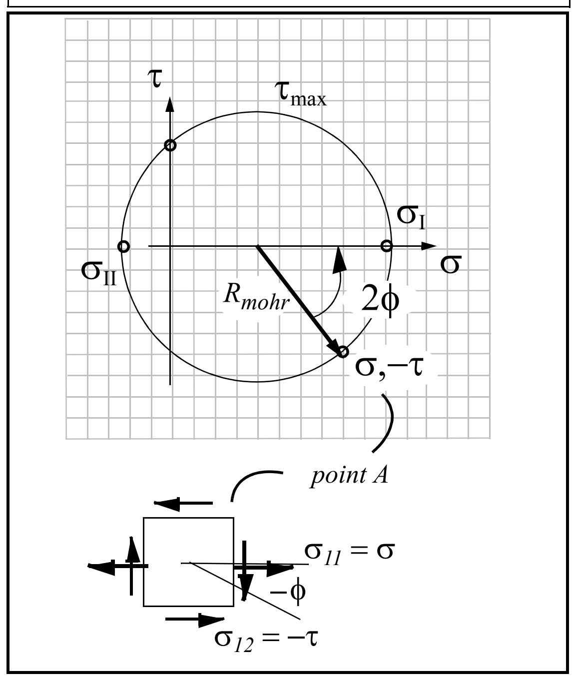
We are concerned with the possibility of the *onset* of plastic flow, of yielding. We apply the Tresca criteria which states that yielding will occur at a point in our material when the maximum shear stress at that point reaches the value of the maximum shear stress in a uniaxial tensile test. This is just one-half the yield stress observed in the tension test.(Top figure below).

Tresca criteria for onset of plastic flow:

$$\tau_{max} = \sigma_{yield}/2$$



Mohr's circle is a graphical representation, NOT OF THE CROSS SECTION OF THE SHAFT OR ANYTHING ELSE THAT COULD BE CONSIDERED TO HAVE MATERIAL SUBSTANCE, but of the mathematical relationships describing the way the components of a (two-dimensional) state of stress at a point change as you change your reference system at the point. Each point along the circumference of the circle gives the values of the normal and shear stress components acting at a different orientation of our reference system.



We are interested in finding the maximum value for the shear stress component - so we can apply the Tresca criteria. That point is at the top (or at the bottom of the circle) Its value equals the radius of the circle:

i.e. $$\tau_{max} = R_{mohr} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + (\tau)^2}$$

Mohr's circle also depicts how much you must rotate your reference system to get to the plane whereupon the shear stress attains this value. It also shows how much you must rotate your reference system to get to the plane(s) of maximum normal stress, the "principle" components of stress at the point, σ_I and σ_{II}.