LECTURE 33

1.060 ENGINEERING MECHANICS II

UNSTEADY FLOW IN OPEN CHANNELS

In Recitation #8 the general equations governing unsteady flows in a prismatic open channel were derived. The consisted of Continuity (Eq 4 in R#8)

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad (1)$$

and Momentum (Eq. 12 in R#8)

$$\underset{(\mathrm{V})}{\frac{\partial V}{\partial t}} + \underset{(\mathrm{IV})}{V \frac{\partial V}{\partial x}} + \underset{(\mathrm{III})}{g \frac{\partial h}{\partial x}} + \underset{(\mathrm{II})}{g \frac{n^2}{R_h^{4/3}} V^2} - \underset{(\mathrm{I})}{g S_o} = 0 \qquad (2)$$

It is instructive to examine the dynamics of the various types of flows one may encounter by considering the terms involved in the Momentum Equation.

<u>Terms I & II</u> is a force balance of gravity, I, and boundary resistance, II, i.e. leading to uniform, steady flow solutions. Since flow is steady, Continuity reduces, to Q = constant. <u>Terms I, II, III & IV</u> includes the effects of non-uniformity by considering also the

pressure forces arising from the free surface not being parallel to the bottom, III, and the convective acceleration resulting from a variation of the velocity due to changes in cross-sectional area, IV. Flow is steady and therefore Q = constant. The flow described by these terms is non-uniform, steady flow, or in our terminology "gradually varied flow. <u>Terms I through V</u> bring in unsteadiness, so Q is no longer constant ($\partial A/\partial t \neq 0$) and wave-type solutions can be expected. Flow is classified as unsteady and non-uniform.

In Recitation #8 we derived a solution for an unsteady non-uniform flow in terms of a wave motion propagating on a current. This solution was obtained by neglecting the shear forces, I, in the dynamic equation for the wave motion. This type of "frictionless" wave is referred to as a "dynamic wave"

It would be prudent to examine the conditions under which our neglect of friction is valid and, at the same time, examine the other extreme, when the dynamics are dominated by frictional effects.

To do this we start by assuming that friction dominates in the momentum equation, i.e. we take

$$S_o \cong S_f = \begin{cases} \dfrac{n^2}{R_h^{4/3}} V^2 = \dfrac{n^2}{A^2 R_h^{4/3}} Q^2 & \text{(Manning)} \\ \dfrac{f}{8gR_h} V^2 = \dfrac{f}{8gA^2R_h} Q^2 & \text{(Darcy-Weisbach)} \end{cases} \tag{3}$$

or

$$Q \approx \begin{cases} \dfrac{AR_h^{2/3}}{n}\sqrt{S_o} & \text{(Manning)} \\ A\sqrt{R_h}\sqrt{\dfrac{8g}{f}}\sqrt{S_o} & \text{(Darcy-Weisbach)} \end{cases} \tag{4}$$

Thus, provided friction dominates to the extent that (3) is $\approx$ valid, we see from (4) that

$$Q = Q\,(\text{channel geometry}) = Q(h) \tag{5}$$

Introducing this in continuity, we have

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = \frac{\partial A}{\partial h}\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial h}\frac{\partial h}{\partial x} = 0 \tag{6}$$

where $\partial A/\partial h = b_s$ = surface width of the channel. Introducing this in (6) we have

$$\frac{\partial h}{\partial t} + \left(\frac{\partial Q/\partial h}{b_s}\right)\frac{\partial h}{\partial x} = 0. \tag{7}$$

For simplicity we assume a wide rectangular channel, for which

$$A = h b_s = h b \quad \text{and} \quad R_h = \frac{A}{b} = h$$

and obtain from (4) [with f taken as constant]

$$\frac{\partial Q}{\partial h} = \begin{cases} \frac{5}{3h} Q \\ \\ \frac{3}{2h} Q \end{cases} \Rightarrow \frac{\partial Q / \partial h}{b_s} = \begin{cases} \frac{5}{3} \frac{Q}{bh} = \frac{5}{3} V \quad (M) \\ \\ \frac{3}{2} \frac{Q}{bh} = \frac{3}{2} V \quad (D\text{-}W) \end{cases}$$

Thus, if we assume that $V = V_0 + v$, with $v \ll V$, then we have from (7) that the governing equation takes the form

$$\frac{\partial h}{\partial t} + c_0 \frac{\partial h}{\partial x} = 0 \qquad (8)$$

where

$$c_0 = \begin{cases} (5/3) V_0 \quad (M) \\ (3/2) V_0 \quad (D\text{-}W) \end{cases} \qquad (9)$$

The solution to (8) with c_0 = constant is in the form of a wave given by

$$h(x,t) = h(x - c_0 t) \qquad (10)$$

i.e. a wave traveling in the +x-direction at a constant velocity, $c_0 = (\frac{3}{2} \text{ or } \frac{5}{3}) V_0$, without change in form.

The solution given by (10) with c_0 given by (9) is known as the "kinematic wave" solution. It is called this because the unsteadiness, i.e. the wave-like behavior, is obtained from the unsteady continuity equation [which is purely kinematic] whereas no unsteady effect is retained in the momentum equation [which represents the dynamics]. In contrast, the solution obtained in Recitation #8, where friction was entirely neglected in the momentum equation, is referred to as the "dynamic wave" solution since unsteadiness is retained both in the continuity and the momentum (dynamic) equations.

The kinematic wave can <u>never</u> propagate in the upstream direction, $c > 0$ always. The dynamic wave can, for subcritical flow, move both in the upstream and the downstream direction. Besides this difference, the respective speeds of propagation differ, for highly subcritical flow, significantly. The dynamic wave speed is $\approx \sqrt{g h_{mo}}$ whereas the kinematic wave speed is $(1.5 \text{ to } 1.67) V_o$, i.e. lower by a factor of $\approx 1.6 \, \mathcal{F}r_o$.

So, when does friction dominate to the extent assumed when the terms (III) through (V) in the momentum equation (2) were neglected? To examine this question we first consider when the pressure term $g\,\partial h/\partial x$ is safely neglected. This would be the case if this term is small relative to one of the terms we retained. Thus, if

$$g\frac{\partial h}{\partial x} \ll g S_0 \Rightarrow \frac{\partial h}{\partial x} \ll S_0$$

neglect of the surface slope term is justified. With the depth changing by an amount Δh over a characteristic distance L, representing the length of the disturbance, we have

$$L \gg \frac{\Delta h}{S_0} \simeq \frac{H_0}{S_0} \qquad (11)$$

Thus, if the disturbance length is several times the distance required for the bottom elevation to change by the characteristic height of the wave, then the wave would behave like the kinematic variety. For $H_0 = 1m$ and $S_0 = 10^{-4}$, we are talking about $L \gg 10^4 m = 10\ km$, in order to neglect $g\,\partial h/\partial x$ in the momentum equation.

But is this length sufficient to neglect all the terms? Let's look at when

$$V\frac{\partial V}{\partial x} = \frac{1}{2}\frac{\partial V^2}{\partial x} \ll g\frac{n^2}{R_h^{4/3}}V_0^2$$

or

$$\frac{\Delta(V^2)}{2L} \ll g\frac{n^2}{R_h^{4/3}}V_0^2$$

Taking $\Delta(V^2)$ conservatively large and equal V_0^2 we obtain

$$L \gg \frac{R_h^{4/3}}{2n^2 g} = \frac{4R_h}{f} \simeq 200 R_h \qquad (12)$$

when $n^2 = (f/8g)R_h^{1/3}$ is used to replace Manning by Darcy-Weisbach friction. This condition, with $R_h \leq h_0$, is generally far less restrictive than (11).

Thus, as a rough guide to whether an unsteady wave motion in an open channel we have

$$\underline{\text{Dynamic}: L \ll \frac{H}{S_0} \ll L \text{ for kinematic wave}}$$

The kinematic wave is used to route floods down rivers, e.g. local heavy rain fall creating a flood condition (that can be of considerable length ~ scale by scale of rain storm or scale of local water shed discharging into the river) that proceeds down the river overtopping levees etc. Magnitude and arrival time of maximum flood is important to forecast to prevent disasters.