1.054/1.541 Mechanics and Design of Concrete Structures (3-0-9)

Design Example

Analysis of Rectangular Slabs using Yield Line Theory

Objective: To investigate the ultimate load of a rectangular slab supported by four fixed edges.

Problem: A reinforced concrete slab (shown in Fig. 1) is supported by four fixed edges. It has a uniform thickness of 8 in., resulting in effective depths in the long direction of 7 in. and in the short direction of 6.5 in. Bottom reinforcement consists of #4 bars at 15 in. centers in each direction and top reinforcement consists of #4 bars at 12 in. in each direction. Material strengths are

Concrete

Uniaxial compressive strength: $f_c^{'} = 4000$ psi;

Steel

Yield stress: $f_y = 60$ ksi.

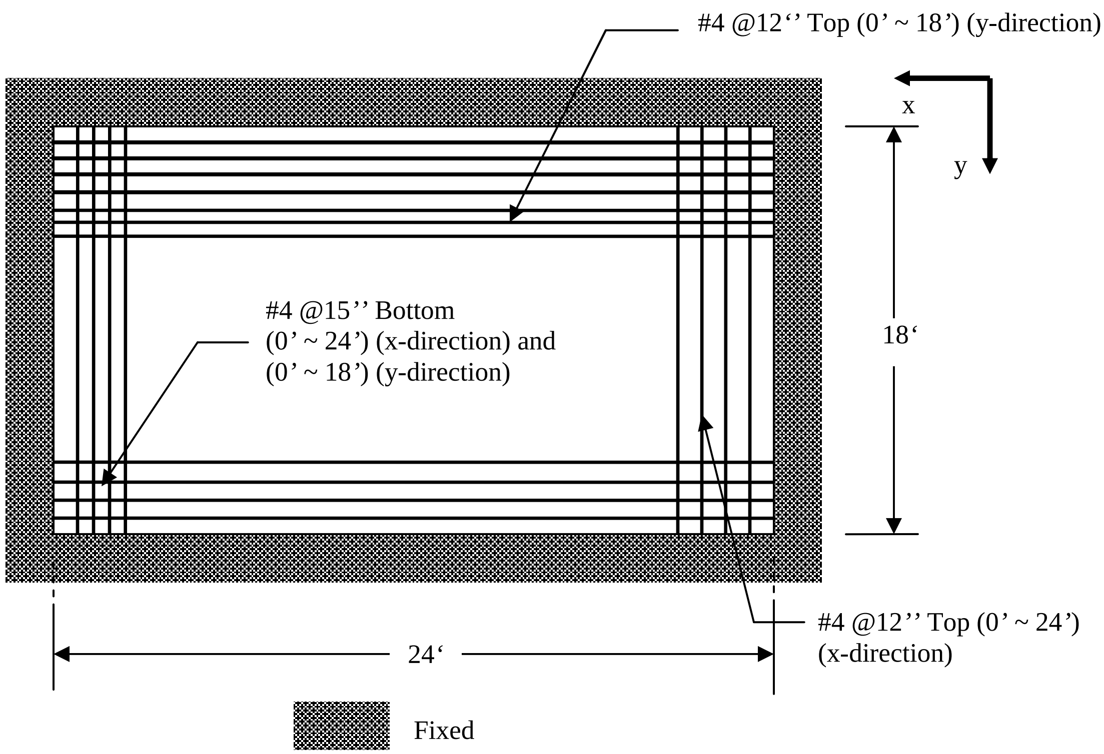


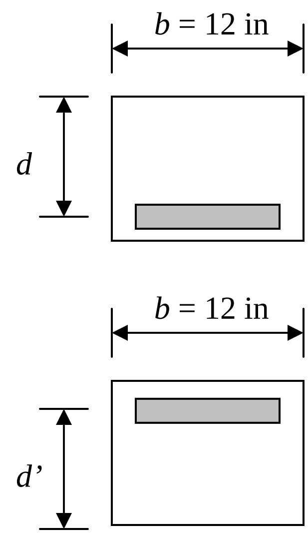
Fig. 1 Reinforced concrete slab and its dimensions

Task: Using the yield line theory method, determine the ultimate load w_u that can be carried by the slab.

[Design Procedures]

Given the information about the slab, shown in Fig. 1, below:

Thickness of the slab: 8 in,
Effective depth in x direction: 7 in,
Effective depth in y direction: 6.5 in,
Notice that *d* = *d'* for each direction.



Reinforcements in both directions:

X direction

Top: #4@12 in (0'~24')
Bottom: #4@15 in (0'~24')

Y direction

Top: #4@12 in (0'~18')
Bottom: #4@15 in (0'~18')

Material strengths:

Concrete

Uniaxial compressive strength: $f_c^{'} = 4000$ psi,

Steel

Yield stress: $f_y = 60$ ksi.

1. Calulation of the moments per unit length in both directions

(1) X direction ($d = d' = 7$ in)

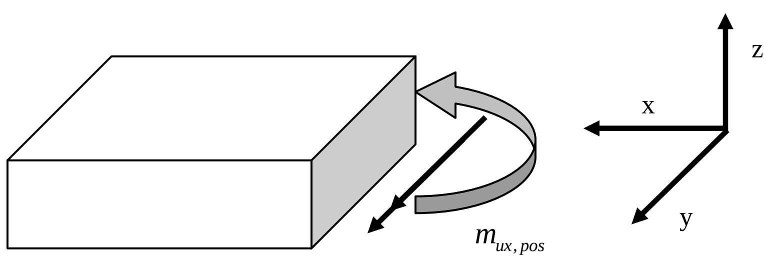


Fig. 2 Positive bending moment in X direction

We have
#4@15 in. for positive (bottom, 0'~18') reinforcement (Fig. 2),
#4@12 in. for negative (top, 0'~18') reinforcement (Fig. 3),

Unit length moments are calculated below.

$\underline{m_{ux,pos}:}$

$$A_s = \frac{12 \text{ in}}{15 \text{ in}} \cdot 0.2 \text{ in}^2 = 0.16 \text{ in}^2$$

$\sum C = \sum T$ provides

$$\Rightarrow 0.85 f_c^{'} ab = A_s f_y \Rightarrow a = \frac{f_y}{0.85 f_c^{'} b} A_s = 1.4706 \cdot 0.16 = 0.235 \text{ in}$$

$$m_{ux,pos} = \phi A_s f_y \left(d - \frac{a}{2} \right) = 0.9 \cdot 0.16 \cdot 60 \cdot \left(7 - \frac{0.235}{2} \right) = 59.46 \text{ kips-in} = \underline{4.95 \text{ kips-ft}}$$

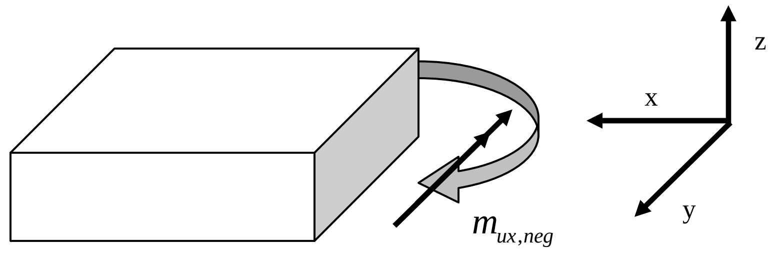


Fig. 3 Negative bending moment in X direction

$\underline{m_{ux,neg}}$ (0'~18'):

$$A_s^{'} = \frac{12 \text{ in}}{12 \text{ in}} \cdot 0.2 \text{ in}^2 = 0.2 \text{ in}^2$$

$\sum C = \sum T$ provides

$$\Rightarrow 0.85 f_c^{'} a^{'} b = A_s^{'} f_y \Rightarrow a^{'} = 1.4706 \cdot 0.2 = 0.294 \text{ in}$$

$$m_{ux,neg} = \phi A_s^{'} f_y \left(d^{'} - \frac{a^{'}}{2} \right) = 0.9 \cdot 0.2 \cdot 60 \cdot \left(7 - \frac{0.294}{2} \right) = 74.01 \text{ kips-in} = \underline{6.17 \text{ kips-ft}}$$

(2) Y direction ($d = d' = 6.5$ in)

We have #4@15 in. for positive (bottom, 0'~24') reinforcement (Fig. 4) and #4@12in. for negative (top, 0'~24') reinforcement (Fig. 5). Calculate the positive and negative moments per unit length respectively.

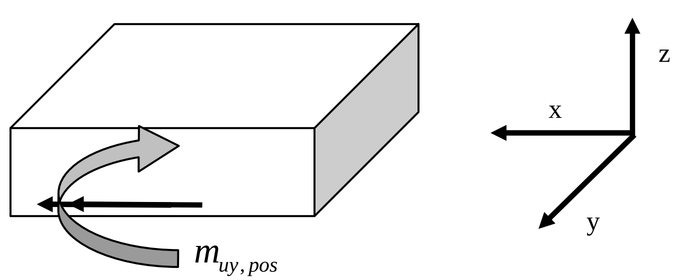


Fig. 4 Positive bending moment in Y direction

$\underline{m_{uy,pos}}$:

$$A_s = \frac{12 \text{ in}}{15 \text{ in}} \cdot 0.2 \text{ in}^2 = 0.16 \text{ in}^2$$

$\sum C = \sum T$ provides

$$\Rightarrow 0.85 f_c^{'} ab = A_s f_y \Rightarrow a = \frac{f_y}{0.85 f_c^{'} b} A_s = \frac{60}{0.85 \cdot 4 \cdot 12} \cdot 0.16 = 1.4706 \cdot 0.16 = 0.235 \text{ in}$$

$$m_{uy,pos} = \phi A_s f_y \left(d - \frac{a}{2} \right) = 0.9 \cdot 0.16 \cdot 60 \cdot \left(6.5 - \frac{0.235}{2} \right) = 55.14 \text{ kips-in} = \underline{4.6 \text{ kips-ft}}$$

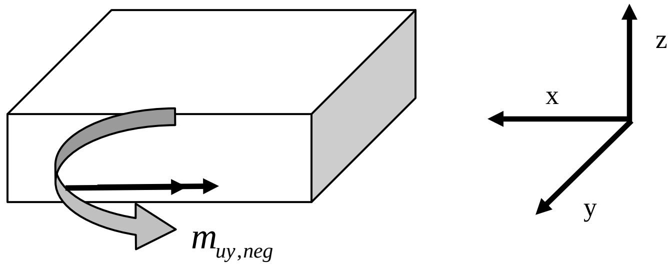


Fig. 5 Negative bending moment in X direction

$\underline{m_{uy,neg}}$:

$$A_s^{'} = \frac{12 \text{ in}}{12 \text{ in}} \cdot 0.2 \text{ in}^2 = 0.2 \text{ in}^2$$

$\sum C = \sum T$ provides

$$\Rightarrow 0.85 f_c^{'} a^{'} b = A_s^{'} f_y \Rightarrow a^{'} = 1.4706 \cdot 0.2 = 0.294 \text{ in}$$

$$m_{uy,neg} = \phi A_s^{'} f_y \left(d^{'} - \frac{a^{'}}{2} \right) = 0.9 \cdot 0.2 \cdot 60 \cdot \left(6.5 - \frac{0.294}{2} \right) = 68.61 \text{ kips-in} = \underline{5.71 \text{ kips-ft}}$$

2. Failure mode and the ultimate load of the slab

   (1) One possible mode is postulated for the slab. Its geometry and associated length and angle definitions are provided in Fig. 6.

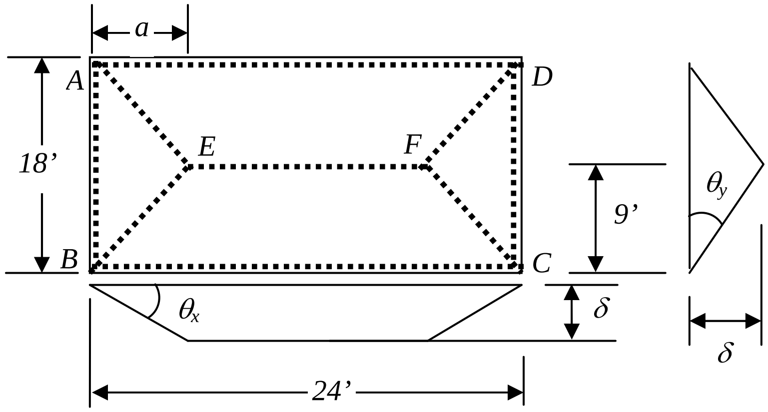


Fig. 6 Postulated failure mode and the associated length and angle definitions

$$\Rightarrow \theta_x = \frac{1}{a},\ \theta_y = \frac{1}{9}$$

Internal work is computed as

Segment	θ_x	θ_y	$m_y\theta_x l_y$	$m_x\theta_y l_x$
AB, CD	$1/a$	0	0	$6.17\cdot\frac{1}{a}\cdot 18$
AD, BC	0	1/9	0	$5.71\cdot\frac{1}{9}\cdot 24$
AE, BE, CF, DF	$1/a$	1/9	$5.71^{*}\cdot\frac{1}{a}\cdot 9$	$4.95^{*}\cdot\frac{1}{9}\cdot a$
EF	0	2/9	0	$4.6\cdot\frac{2}{9}\cdot(24-2a)$

[*: Use 5.71 and 4.95 kips-in to be conservative although the moment varies along these yield lines.]

$$\sum W_{\text{int}} = 2\left[\frac{111.06}{a}+15.23\right]+4\left[\frac{51.39}{a}+0.55a\right]+24.48-2.04a$$

$$= \frac{427.68}{a}+54.94+0.16a$$

External work is computed as

Segment	Area	δ	$w\cdot A\cdot\delta$
ABE, CDF	$\frac{18\cdot a}{2}$	1/3	$3wa$
BCFE, ADFE	$9a$; $(24-2a)\cdot 9 = 216-18a$	1/3; 1/2	$3wa+108w-9wa = 108w-6wa$

$$\sum W_{\text{ext}} = 2\left[3wa+108w-6wa\right] = 216w-6wa = w(216-a)$$

$$\because \sum W_{\text{int}} = \sum W_{\text{ext}}$$

$$\therefore \Rightarrow \frac{427.68}{a}+54.94+0.16a = w(216-6a)$$

$$\Rightarrow w = \frac{427.68+54.94a+0.16a^2}{216a-6a^2}$$

For minimum w, $\frac{dw}{da}=0$ and $\frac{d^2w}{da^2}>0$

$\frac{dw}{da}=0$ provides

$$\frac{dw}{da}=\frac{\left[\left(54.94+0.32a\right)\left(216a-6a^2\right)-\left(427.68+54.94a+0.16a^2\right)\cdot\left(216-12a\right)\right]}{\left(216a-6a^2\right)^2}=0$$

It yields to

$$11867a-329.64a^2+69.12a^2-1.92a^3-92378.88+5132.16a-11867a+659.28a^2$$
$$-34.56a^2+1.92a^3=0$$

$$\Rightarrow 364.2a^2+5132.16a-92378.88=0$$
$$\Rightarrow a^2+14.09a-253.65=0$$

and a = 10.37 in

$$\because w=\frac{427.68+54.94a+0.16a^2}{216a-6a^2}$$

$$\therefore w_u=0.636\frac{\text{kips}}{\text{ft}^2}$$

Therefore the ultimate load this rectangular slab can carry is 0.636 kips/ft^2.