# 5

# Indeterminate Systems

The key to resolving our predicament, when faced with a statically indeterminate problem - one in which the equations of static equilibrium do not suffice to determine a unique solution - lies in opening up our field of view to consider the *displacements of points* in the structure and the *deformation of its members*. This introduces new variables, a new genera of flora and fauna, into our landscape; for the truss structure the species of *node displacements* and the related species of *uniaxial member strains* must be engaged. For the frame structure made up of beam elements, we must consider the *slope of the displacement* and the related *curvature of the beam* at any point along its length. For the shaft in torsion we must consider the rotation of one cross section relative to another.

Displacements you already know about from your basic course in physics – from the section on Kinematics within the chapter on Newtonian Mechanics. Displacement is a vector quantity, like force, like velocity; it has a magnitude and direction. In Kinematics, it tracks the movement of a physical point from some location at time $t$ to its location at a subsequent time, say $t + \delta t$, where the term $\delta t$ indicates a small time increment. Here, in this text, the displacement vector will, most often, represent the movement of a physical point of a structure from *its position in the undeformed state of the structure to its position in its deformed state*, from the structure's unloaded configuration to its configuration under load.

These displacements will generally be small relative to some nominal length of the structure. Note that previously, in applying the laws of static equilibrium, we made the tacit assumption that *displacements were so small we effectively took them as zero;* that is, we applied the laws of equilibrium to the *undeformed body.*[1] There is nothing inconsistent in what we did there with the tack we take now as long as we restrict our attention to small displacements. That is, our equilibrium equations taken with respect to the undeformed configuration remain valid even as we admit that the structure deforms.

Although *small* in this respect, the small displacement of one point *relative to* the small displacement of another point in the deformation of a structural member can engender large internal forces and stresses.

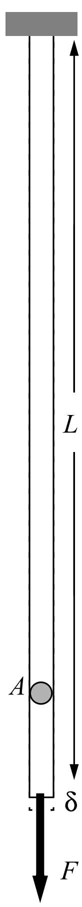
In a first part of this chapter, we do a series of exercises - some simple, others more complex - but all involving only one or two degrees of freedom; that is, they all concern systems whose deformed configuration is defined by but one or two displacements (and/or rotations). In the final part of this chapter, we consider

1. The one exception is the introductory exercise where we allowed the two bar linkage to "snap through"; in that case we wrote equilibrium with respect to the deformed configuration.

indeterminate truss structures - systems which may have many degrees of freedom. In subsequent chapters we go on to resolve the indeterminacy in our study of the shear stresses within a shaft in torsion and in our study of the normal and shear stresses within a beam in bending.

## 5.1 Resolving indeterminacy: Some Simple Systems.

If we admit displacement variables into our field of view, then we must necessarily learn how these are related to the forces which produce or are engendered by them. We must know how force relates to displacement. Force-displacement, or *constitutive* relations, are one of three sets of relations upon which the analysis of indeterminate systems is built. The requirements of *force and moment equilibrium* make up a second set; *compatibility of deformation* is the third.




### A Word about Constitutive Relations

You are familiar with one such constitutive relationship, namely that between the force and displacement of a spring, usually a linear spring. $F = k \cdot \delta$ says that the force $F$ varies linearly with the displacement $\delta$. The spring constant (of proportionality) $k$ has the *dimensions* of force/length. It's particular *units* might be pounds/inch, or Newtons/millimeter, or kilo-newtons/meter.

Your vision of a spring is probably that of a coil spring - like the kind you might encounter in a children's playground, supporting a small horse. Or you might picture the heavier springs that might have been part of the undercarriage of your grandfather's automobile. These are real-world examples of linear springs.

But there are other kinds that don't look like coils at all. A rubber band behaves like a spring; it, however, does not behave linearly once you stretch it an appreciable amount. Likewise an aluminum or steel rod when stretched behaves like a spring and in this case behaves linearly over a useful range - but you won't see the extension unless you have superhuman eyesight.

For example, the picture at the left is meant to represent a rod, made of an aluminum alloy, drawn to full scale. It's length is $L = 4$ inches, its cross-sectional area $A = 0.01$ square inches. If we apply a force, $F$, to the free end as shown, the rod will stretch, the end will move downward just as a coil spring would. And, for small deflections, $\delta$, if we took measurements in the lab, plotted force versus displacement, then measured the slope of what appears to be a straight line, we would have:

$$F = k \cdot \delta \qquad \text{where} \qquad k = 25{,}000 \text{ lb/inch}$$

This says that if we apply a force of 25,000 pounds, we will see an end displacement of 1.0 inch. You, however, will find that you can not do so.

The reason is that if you tried to apply a weight of this magnitude (more than 10 tons!) the rod would stretch more and more like a soft plastic. It would *yield* and fail. So there are limits to the loads we can apply to materials. That limit is a characteristic (and conventional) property of the material. For this particular aluminum alloy, the rod would fail at an axial stress of

$\sigma_{yield} = 60{,}000 \text{ psi}$ or at a force level $F = 600 \text{ pounds}$ factoring in the area of 0.01 square inches.

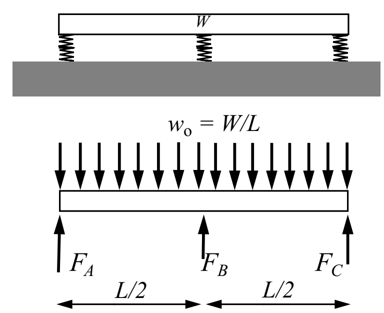
Note that at this load level, the end displacement, figured from the experimentally established stiffness relation, is $\delta = 0.024 \text{ inches}$ (can you see that?) And thus the ratio $\delta/L$ is but 0.006. This is what we mean by small displacements. This is what we mean by linear behavior (only up to a point - in this case -the yield stress). This is the domain within which engineers design their structures (for the most part).

We take this as the way force is related to the displacement of individual structural elements in the exercises that follow[2].

**Exercise 5.1**

*A massive stone block of weight W and uniform in cross section over its length L is supported at its ends and at its midpoint by three linear springs. Assuming the block a rigid body[3], construct expressions for the forces acting in the springs in terms of the weight of the block.*

The figure shows the block resting on three linear springs. The weight per unit length we designate by $w_o = W/L$.




In the same figure, we show a free-body diagram. The forces in the spring, taken as compressive, push up on the beam in reaction to the distributed load. Force equilibrium in the vertical direction gives:

2. We will have more to say about constitutive relations of a more general kind in a subsequent chapter.
3. The word *rigid* comes to the fore now that we consider the deformations and displacements of extended bodies. *Rigid* means that there is no, *absolutely no relative displacement of any two, arbitrarily chosen points in the body* when the body is loaded. Of course, this is all relative in another sense. There is always some relative displacement of points in each, every and all bodies; *a rigid body* is as much an abstraction as *a frictionless pin*. But in many problems, the relative displacements of points of some one body or subsystem may be assumed small relative to the relative displacements of another body. In this exercise we are claiming that the block of stone is rigid, the springs are not, i.e., they deform.

$$F_A + F_B + F_C - W = 0$$

While moment equilibrium, summing moments about the left end, *A*, taking counter-clockwise as positive, gives:

$$\sum_A M = F_B \cdot L/2 + F_C \cdot L - W \cdot L/2 = 0$$

The problem is *indeterminate*: Given the length *L* and the weight *W*, we have but *two* equations for the *three* unknown forces, the three compressive forces in the springs.

Now, indeterminacy *does not* mean we can not find a solution. What it *does* mean is that *we can not find* a *single, unambiguous, unique solution* for each of the three forces. That is what indeterminate means. We can find solutions - *too many* solutions; the problem is that we do not have sufficient information, e.g., enough equations, to fix which of the many solutions that satisfy equilibrium is the right one[4].

*Indeterminate solution (to equilibrium alone) #1*

For example, we might take $F_B = 0$, which in effect says we remove the spring support at the middle. Then for equilibrium we must subsequently have $F_A = F_C = W/2$ This is a solution to equilibrium.

*Indeterminate solution (to equilibrium alone) #2*

Alternatively, we might require that $F_A = F_C$; in effect adding a third equation to our system. With this we find from moment equilibrium that $F_A = F_C = W/3$ and so from force equilibrium $F_B = W/3$ This too is a solution.

*Indeterminate solution (to equilibrium alone) #n, n=1,2,......*

We can fabricate many different solutions in this way, an infinite number. For example, we might arbitrarily take $F_B = W/n$, where n = 1,2,....then from the two requirements for equilibrium find the other two spring forces. (Try it)!

Notice in the above that we have not said one word about the displacements of the rigid block nor a word about the springs, their stiffness, whether they are linear springs or non-linear springs. Now we do so. Now we really solve the indeterminate problem, setting three or four different scenarios, each defined by a different choice for the relative stiffness of the springs. In all cases, we will assume the springs are linear.

4. We say the equations of equilibrium are *necessary* but not *sufficient* to produce a solution.

*Full Indeterminate solution, Scenario #1*

In this first scenario, at the start, we assume also that they have equal stiffness.

We set

$$F_A = k \cdot \delta_A$$

$$F_B = k \cdot \delta_B$$ where $\delta_A$, $\delta_B$, and $\delta_C$ are the displacements of the springs,

$$F_C = k \cdot \delta_C$$

taken positive downward since the spring forces were taken positive in compression[5]. The spring constants are all equal. These are the required *constitutive relations*.

Now *compatibility of deformation*: The question is, how are the three displacements related. Clearly they must be related; we can not choose them independently one from another, e.g., taking the displacements of the end springs as downward and the displacement of the midpoint as upwards. This could only be the case if the block had fractured into pieces. No, this can't be. We insist on compatibility of deformation.

Here we confront the same situation faced by Buridan's ass, that is, the situation to the left appears no different from the situation to the right so, "from symmetry" we claim there is no sufficient reason why the block should tip to the left or to the right. It must remain level[6].

In this case, the displacements are all equal.

$$\delta_A = \delta_B = \delta_C$$

This is our *compatibility* equation.

So, in this case, from the constitutive relations, the spring forces are all equal.

So, in this case,

$$F_A = F_B = F_C = W/3$$

*Full Indeterminate solution, Scenario #2*

In this second scenario, we assume the two springs at the end have the same stiffness, k, while the stiffness of the spring at mid-span is different. We set $k_B = \alpha k$ so our constitutive relations may be written

$$F_A = k \cdot \delta_A$$

$$F_B = \alpha k \cdot \delta_B$$ where the non-dimensional parameter $\alpha$ can take on any posi-

$$F_C = k \cdot \delta_C$$

tive value within the range 0 to very, very large.

---

5. We must be careful here; a positive force must correspond to a positive displacement.
6. Note that this would not be the case if the spring constants were chosen so as to destroy the symmetry, e.g., if $k_A > k_B > k_C$.

Notice again we have symmetry: There is still no reason why the block should tip to the left or to the right! So again, the three displacements must be equal.

$$\delta_A = \delta_B = \delta_C = \delta$$

The constitutive relations then say that the forces in the two springs at the end are equal, say = $F$ and that the force in the spring at mid span is $\alpha F$.

With this, force equilibrium gives

$$F_A + F_B + F_C = W \qquad \text{i.e.,} \qquad (2+\alpha) \cdot k \cdot \delta = W$$

So, in this scenario,

$$F_A = F_C = \frac{W}{(2+\alpha)} \qquad \text{and} \qquad F_B = \frac{\alpha \cdot W}{(2+\alpha)}$$

- Note that if we set $\alpha$=0, in effect removing the middle support, we obtain what we obtained before - *indeterminate solution (to equilibrium) #1.*
- Note that if we set $\alpha$=1.0, so that all three springs have the same stiffness, we obtain what we obtained before - *full indeterminate solution, Scenario #1.*
- Note that if we let $\alpha$ be a very, very large number, then the forces in the springs at the ends become very, very small relative to the force in the spring at mid-span. In effect we have removed them. (We leave the stability of this situation to a later chapter).

*Full Indeterminate solution, Scenario #3*

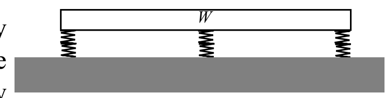
We can play around with the relative values of the stiffness of the three springs all day if we so choose. While not wanting to spend all day in this way, we should at least consider one scenario in which we loose the symmetry, in which case the springs experience different deformations.

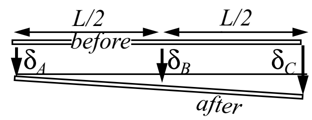
Let us take the stiffness of the spring at the left end equal to the stiffness of the spring at midspan, but now set the stiffness of the spring at the right equal to but a fraction of the former;

That is we take

$$F_A = k \cdot \delta_A$$
$$F_B = k \cdot \delta_B$$
$$F_C = \alpha k \cdot \delta_C$$

Clearly we have lost our symmetry. We need to reconsider compatibility of deformation, considering how the displacements of the three springs must be related.




The figure at the right is *not* a free body diagram. It is a new diagram, simpler in many respects than a free body diagram. It is a picture of the displaced structure, rather a picture of how it might possibly displace.

"Possibilities" are limited by our requirement that the block remain all in one piece and *rigid*. This means that the points representing the locations of the ends of the springs, at their junctions with the block, in the displaced state *must all lie on a straight line*.

The figure shows the *before* and *after* loading states of the system.

There is now a rotation of the block as well as a vertical displacement[7]. Now, we know that it takes only two points to define a straight line. So say we pick $\delta_A$ and $\delta_B$ and pass a line through the two points. Then, if we extend the line to the length of the block, the intersection of a vertical line drawn through the end at *C* in the undeflected state and this extended line will define the displacement $\delta_C$.

In fact, from the geometry of this displaced state, chanting "...similar triangles...", we can claim

$$\frac{(\delta_B - \delta_A)}{(L/2)} = \frac{(\delta_C - \delta_A)}{L} \qquad \text{or} \qquad \delta_B = \frac{1}{2} \cdot (\delta_C + \delta_A)$$

This second equation shows that the midspan displacement is the mean of the two end displacements.

This is our *compatibility* condition. It holds *irrespective of our choice of spring stiffness*. It is an *independent requirement*, independent of equilibrium as well. It is a consequence of our assumption that the block is rigid.

Now, with our assumed constitutive relations, we find that the forces in the springs may be written in terms of the displacements as follows.

$$F_A = k \cdot \delta_A$$

$$F_B = k \cdot \frac{1}{2} \cdot (\delta_C + \delta_A)$$ where we have eliminated $\delta_B$ from our story.

$$F_C = \alpha k \cdot \delta_C$$

Equilibrium, expressed in terms of the two displacements, $\delta_A$ and $\delta_C$. gives:

$$\delta_A + \frac{1}{2} \cdot (\delta_C + \delta_A) + \alpha\delta_C = W/k \qquad \text{and} \qquad \frac{(\delta_C + \delta_A)}{4} + \alpha\delta_C = W/(2k)$$

7. We say the system now has *two degrees of freedom*.

The solution to these is:

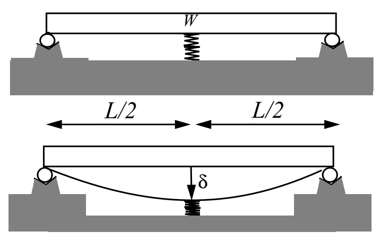
$$\delta_A = \frac{2W}{k} \cdot \frac{\alpha}{(1+5\alpha)} \qquad \delta_B = \frac{W}{k} \cdot \frac{(1+\alpha)}{(1+5\alpha)} \qquad \text{and} \qquad \delta_C = \frac{2W}{k} \cdot \frac{1}{(1+5\alpha)}$$

- Note that if we take $\alpha = 1.0$, we again recover the symmetric solution $\delta_A = \delta_B = \delta_C$ and $F_A = F_B = F_C = W/3$
- Note that if we take $\alpha = 0$ we obtain the interesting result $\delta_A = 0$ $\delta_B = W/k$ $\delta_C = 2 \cdot W/k$ which means that the block pivots about the left end. And the midspan spring carries all of the weight of the block! $F_A = F_C = 0$ $F_B = W$
- And if we let $\alpha$ get very, very large...(see the problem at the end of the chapter.

*Full Indeterminate solution, Scenario #4*

As a final variation on this problem, we relinquish our claim that the block is rigid. Say it is not made of stone, but of some more flexible, structural material such as aluminum, or steel, or wood, or even glass. We still assume that the weight is uniformly distributed over its length.

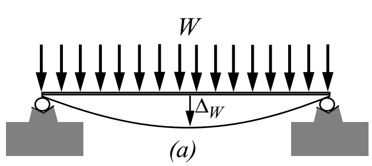
We will, however, assume the spring stiffness are of a special form in order to obtain a relatively simple problem formulation and resolution. We take the end springs as infinitely stiff, as rigid. They deflect not at all. In effect we support the block at its ends by pins. The stiffness of the spring at midspan we take as *k*.



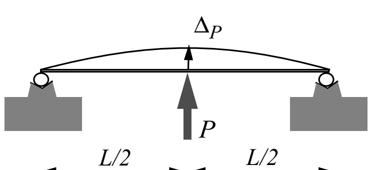

Our picture of the geometry of deformation must be redrawn to allow for the relative displacement of points, any two points, in the block.

We again, assuming the block is uniform along its length, can claim symmetry. We sketch the deflected shape accordingly.

Equilibrium remains as before. But now we must be concerned with the constitutive relations for the beam!




(a)




(b)

Forget the spring for a moment. Picture the beam as subjected to two different loadings: The first, a uniformly distributed load, figure (a); the second, a concentrated load P at midspan, as shown in figure (b), due to the presence of the spring.

Let the deflection at midspan due to first loading condition, *W,* uniformly distributed load, be designated by $\Delta_W$. Take it from me that we can write

$$W = k_W \cdot \Delta_W$$

that is, the deflection grows linearly with the weight[8]. Here $k_W$ is the stiffness, relating the midspan deflection to the total weight of the block.

Let the deflection at midspan due to the second loading condition, the concentrated load, P, be designated by $\Delta_P$. Take it from me that we can write $P = k_P \cdot \Delta_P$ where we will, in time, identify P as the force due to the compression of the spring.

Now, for *compatibility of deformation*, the actual deflection at midspan will be the difference of these two deflections: If we take downward as positive we have

$$\delta = \Delta_W - \Delta_P$$

where δ is the net downward displacement at midspan and hence, the actual compressive displacement of the spring. Putting this in terms of the spring force and the weight W, and the force P, we have:

$${}_B/k = W/k_W - P/k_P \quad \textit{But P is just } F_B\text{. So we have} \quad {}_B/k = W/k_W - F_B/k_P$$

We solve this now for the force in the spring in terms of the total weight, *W*, which we take as given and obtain

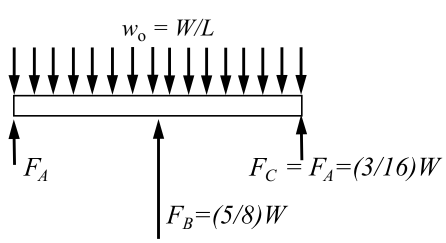
$$F_B = \frac{k_P \cdot k}{k_W} \cdot \frac{W}{(k + k_P)}$$

This simplifies if you accept the fact that the ratio of the stiffness, $k_p/k_W$, is known. Take it from me that this ratio is 5/8.

With this we can write $$F_B = (5/8) \cdot \frac{W}{(1 + k_P/k)}$$

Then from force equilibrium we obtain: $$F_A = F_C = \frac{1}{2} \cdot \frac{(3/8 + k_P/k)}{(1 + k_P/k)} \cdot W$$

- Note that if we let the stiffness, k, of the spring get very, very large, we get $F_A = F_C = (3/16) \cdot W$ *while* $F_B = (5/8) \cdot W$ In effect, we have replaced the midspan spring with a rigid, pin support and these are the reaction forces at the supports.




8. We study, and construct expressions for, the displacement distribution of beams due to various loading conditions in a later chapter.

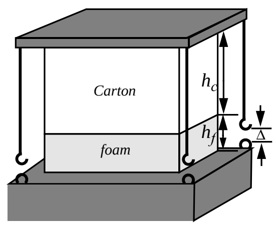
- Note how, knowing the reaction forces, we could go on and draw the shear-force and bending moment diagram.

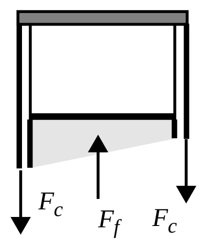
That's enough variations on single problem. We turn now to a second exercise, an indeterminate problem again, to see the power of our three principles of analysis.

**Exercise 5.2**

*A rigid carton, carrying fragile contents (of negligible weight), rests on a block of foam and is restrained by four elastic cords which hold it fast to a truck-bed during transit. Each cord has a spring stiffness* $k_{cord}$ *= 25 lb/in.; the foam has eight times the spring stiffness,* $k_{foam}$ *= 400 lb/in.*

*The gap Δ, in the undeformed state, i.e., when the cords hang free, is 1.0 in.*[9] *Show that when the carton is held down by the four cords that each of the cords experiences a tensile force of 20 lb.*




We begin by making a cut through the body to get at the internal forces in the cords and in the foam. We imagine the cords hooked to the floor; the system in the deformed state. We cut through the foam and the cords at some quite arbitrary distance up from the floor. Our isolation is shown at the right.

Force equilibrium gives, noting there are four cords to take into account:

$$F_f - 4F_c = 0$$

Here we model the system as capable of motion in the vertical direction only. The internal reactive force in the foam is taken as uniformly distributed across the cut. $F_f$ is the resultant of this distribution. Consistent with this, the foam, like the cords, is taken as a uniaxial truss member, like a linear spring.

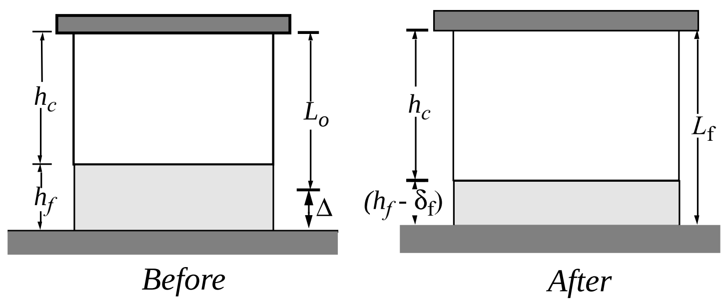
**Observe that the foam will compress, the cords will extend**. Note that I seemingly violate my convention for assumed direction of positive truss member forces in that I take a *compressive* force in the foam as positive. I could argue that this is not a truss; but, no, the real reason for proceeding in this way is to make full use of our physical insight in illustrating the new requirement of compatibility of deformation. We can be quite confident that the foam will compress and the

9. It is not necessary to state that we neglect the weight of the carton if we work from the deformed state with the foam deflected some due to the weight alone. This is ok as long as the relationship between force and deflection is linear which we assume to be the case.

cords extend. In other instances to come, the sign of the internal forces will **not** be so clear. Careful attention must then be given to the convention we adopt for the positive directions of *displacements*, as well as forces. Note also that there exists *no* externally applied forces yet internal forces exist and must satisfy equilibrium. We call this kind of system of internal forces *self-equilibrating*.

We see we have but **one equation** of equilibrium, **yet two unknowns**, the internal forces $F_c$ and $F_f$. The problem is *statically, or equilibrium indeterminate*.

We now call upon the new requirement of *Compatibility of deformation t*o generate another required relationship. In this we designate the compression of the foam $\delta_f$; its units will be *inches*. We designate the extension of the cords $\delta_c$; it too will be measured in *inches*.




*Compatibility of deformation* is a statement relating these two measures. In fact their sum must be, $\Delta$, the original gap. We construct this statement as follows:

The original length of the cord is

$L_o = h_f + h_c - \Delta$ while its final length is $L_f = (h_f - \delta_f) + h_c$

The *extension* of the cord is the difference of these: $\delta_c = L_f - L_O = -\delta_f + \Delta$

*Compatibility of Deformation* then requires

$$\boxed{\delta_c + \delta_f = \Delta}$$

Only if this is true will our structure remain *all together now* as it was before fastening down.

Here is **a second equation** but look, we have introduced **two more unknowns**, the compression of the foam and the extension of the cords. It looks like we are making matters worse! Something more must be added, namely we must relate the internal forces that appear in equilibrium to the deformations that appear in compatibility. This is done through two *constitutive equations*, equations whose form and factors depend upon the material out of which the cord and foam are *constituted*. In this example we have modeled both the foam and the cords as *linear springs*. That is we write

$$F_c = k_c \cdot \delta_c \quad \text{where} \quad k_c = 25(lb/in)$$
$$\text{and}$$
$$F_f = k_f \cdot \delta_f \quad \text{where} \quad k_f = 400(lb/in)$$

These last are **two more equations**, but **no more unknowns**. Summing up we see we now have four *linearly independent* equations for the four unknowns, — the two internal forces and the two measure of deformation.

There are various ways to solve this set of equations; I first write $\delta_f$ in terms of $\delta_c$ using compatibility, i.e. $\delta_f = \Delta - \delta_c$ then express both unknown forces in terms of $\delta_c$.

$$F_c = k_c \cdot \delta_c \qquad and \qquad F_f = k \cdot (\Delta - \delta_c)$$

Equilibrium then yields a single equation for the extension of the cords, namely

$$k_f \cdot (\Delta - \delta_c) - 4k\delta_c = 0 \quad \text{so} \quad \delta_c = \Delta \cdot \left[\frac{k_f}{4k_c + k_f}\right]$$

and we find the tension in the cords, $F_c$ to be:

$$F_c = k_c \cdot \delta_c = 20 \text{ lb.}$$

The compressive force in the foam is four times this, namely 80 *lb*, since there are four cords. Finally, we find that the extension of the cords and the compression of the foam are

$$\delta_c = 0.8 \text{ in.} \qquad and \qquad \delta_f = 0.2 \text{ in.}$$

which sum to the original gap, $\Delta$.

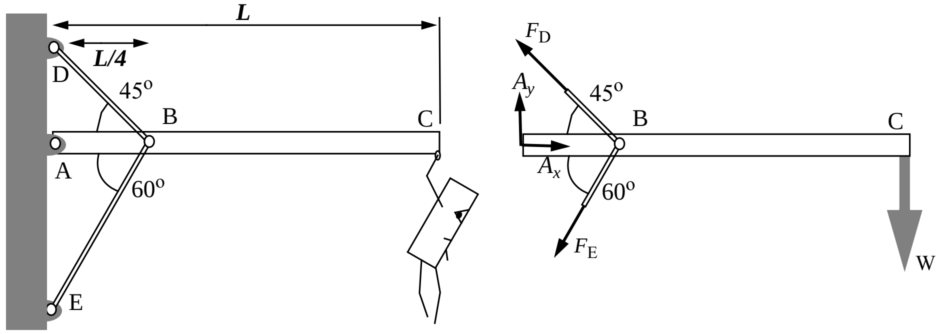
This simple exercise[10] captures all of the major features of the solution of statically indeterminate problems. We see that we must contend with **three requirements: Static Equilibrium, Compatibility of Deformation, and Constitutive Relations**. A less fancy phrasing for the latter is *Force-Deformation Equations*. We turn now to a third exercise which includes truss members under uniaxial loading.

---

10. *Simplicity* is not meant to imply that the exercise is not without practical importance or that it is a simple matter to conjure up all the required relationships: If I were to throw in a little dash of the dynamics of a single degree of freedom, Physics I, Differential Equations, mass-spring system I could start designing cord-foam support systems for the safe transport of fragile equipment over bumpy roads. More to come on this score.

**Exercise 5.3**

*I know that the tip deflection at the end C of the structure — made of a rigid beam ABC of length L= 4m, and two 1020CR steel support struts, DB and EB, each of cross sectional area A and intersecting at a= L/4 — when supporting an individual weighing 800 Newtons is 0.5mm. What if I suspend more individuals of the same weight from the point C; when will the structure collapse?*




Here is a problem statement which, when you approach the punch line, prompts you to suspect the author intends to ask some ridiculous question, e.g., "What time is it in Chicago?" No matter. We know that if it's in this textbook it is going to require a free-body diagram, application of the requirements of static equilibrium, and now, compatibility of deformation and constitutive equations. So we proceed. I start with equilibrium, isolating the *rigid* bar, *ABC*.

**Force Equilibrium:**

$$A_x - F_D \cdot cos45^o - F_E \cdot cos60^o = 0$$

$$A_y + F_D \cdot sin45^o - F_E \cdot sin60^o - W = 0$$

Moment equilibrium (positive ccw), about point[11] *A* yields

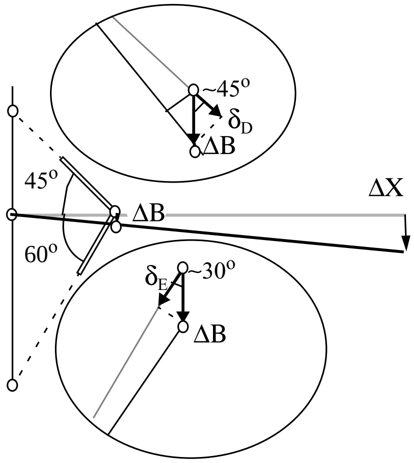
$$F_D \cdot sin45^o \cdot (L/4) - F_E \cdot sin60^o \cdot (L/4) - WL = 0$$

These are three equations for the four unknowns, $A_x$, $A_y$, $F_D$, and $F_E$. The structure is redundant. We could remove either the top or the bottom strut and the remaining structure would support an end load – not as great an end load but still some significant value.

---

11. Member *ABC* is *not* a two-force member even though it shows frictionless pins at *A, B* and *C*. In fact it is not a two-force member because it is a three-force member– three forces act at the three pins ($F_D$ and $F_E$ may be thought of as equivalent to a single resultant acting at *B*). The member must also support an internal bending moment, i.e., over the region *BC* it acts much like a cantilever beam. Note that, while there can be no couple acting at the interface of the frictionless pin and the beam at *B*, there is a bending moment internal to the beam at a section cut through the beam at this point. If you can read this and read it correctly you are mastering the language.

**Compatibility of Deformation:**




The deformations of member *BD* and member *BE* are related. How they relate is not obvious. We draw a picture, attempting to show the motion of the system from the undeformed state (*W*=0) to the deformed state, then relate the member deformations to the displacement of point *B*.

I have let $\Delta_B$ represent the *vertical* displacement of point *B* and $\Delta_C$ the vertical displacement at the tip of the *rigid* beam. Because I have said that member *ABC* is *rigid*, there is no horizontal displacement of point *B* or at least none that matters. If the member is elastic, the horizontal displacement should be taken into account in relating the deformations of the two struts. When the member is rigid, there **is** a horizontal displacement of *B* but for *small vertical displacements*, $\Delta_B$, the horizontal displacement is *second order*. For example, if $\Delta_B/L$ is of order $10^{-1}$, then the horizontal displacement is of order $10^{-2}$.

Shown above the full structure is an exploded view of the vertical displacement $\Delta_B$ and its relationship to the deformation of member *DB*, the extension $\delta_D$. From this figure I take

$$\delta_D = \Delta_B \cdot cos45^o = (\sqrt{2}/2)\Delta_B$$

Shown at the bottom is an exploded view of the vertical displacement and its relationship to the deformation of member *EB*. Taking the measures of deformation as positive in extension, consistent with our convention of taking the member forces as positive in tension, and noting that member *EB* will be in compression, we have

$$\delta_E = -\Delta_B \cdot cos30^o = -(\sqrt{3}/2)\Delta_B$$

These two equations relate the deformations of the two struts through the variable $\Delta_B$. We can read them as saying that, for small deflections and rotations, **the extension or contraction of the member is equal to the projection of the displacement vector upon the member**.

Only if these equations are satisfied are these deformations compatible; only then will the two members remain together, joined at point *B*. This is then our requirement of **compatibility of deformation.**

**Constitutive Equations:**

These are the simplest to write out. We assume the struts are both operating in the elastic region. We have

$$\sigma_{DB} = E \cdot \varepsilon_{DB} \qquad \text{or} \qquad F_D/A = E \cdot \delta_D/( \ \ 2 \cdot L/4)$$

*and*

$$\sigma_{EB} = E \cdot \varepsilon_{EB} \qquad \text{or} \qquad F_E/A = E \cdot \delta_{DE}/(L/2)$$

where I have used the geometry to figure the lengths of the two struts.

Now let's go back and see what was given, what was wanted. We clearly are interested in the forces in the two struts, the two *F*'*s*; more precisely, we are interested in the stresses engendered by the end load *W*, for if either of these stresses reaches the *yield strength* for 1020CR steel we leave the elastic region and must consider the possibility of collapse of our structure. These forces, in turn, depend upon the member deformations, the δ's which, in turn depend upon the vertical deflection at *B*, $\Delta_B$.

We can think of the problem, then, as one in which there are five unknowns. We see that we have *seven* equations available, but note we have the horizontal and vertical components of the reaction force at *A* as unknowns too, so everything is in order. In fact we only need work with five of the seven equations because $A_x$ and $A_y$ appear only in the two, force equilibrium equations. The wise choice of point *A* as our reference point for moment equilibrium enables us to proceed without worrying about these two relations [12] I first express the member forces in terms of $\Delta_B$ using the constitutive and compatibility relations. I obtain

$$F_D = 2 \cdot (AE/L) \cdot \Delta_B \qquad \text{and} \qquad F_E = - \ \ 3 \cdot (AE/L) \cdot \Delta_B$$

where the negative sign indicates that member *EB* is in compression. Note that the magnitude of the tensile load in *DB* is greater than the compression in member *EB*. Now substituting these for the forces as they appear in the equation of moment equilibrium I obtain the following relationship between the end load *W* and the vertical displacement of point *B,* namely:

$$W = \left(\frac{3 + 2 \ \ 2}{8}\right) \cdot (AE/L) \cdot \Delta_B$$

This relationship is worth a few words: It relates the vertical force, *W,* at the point *C*, at the end of the rigid beam, to the vertical displacement, $\Delta_B$,at *another* point in the structure. The factor of proportionality can be read as a stiffness, k, like that of a linear spring. Note that the dimensions of the factor, (AE/L) are just force per length.

---

12. If we were concerned, as perhaps we should be, with the integrity of the fastener at *A* we would solve these two equations to determine the reaction force.

Now I can use this, and the observation that when $W = 800N$, the tip deflection is 0.5mm to obtain an expression for the factor $(AE/L)$. But first I have to relate $\Delta_B$ to the tip deflection $\Delta_C = .5$mm. For this I return to my sketch of the deformed geometry and note, if (and only if) *ABC* is *rigid* then the two deflections are related, "through similar triangles" by $\Delta_B = \Delta_C / 4$

This then yields

$$(AE/L) = \frac{8W}{\left[\left(\frac{\Delta_C}{4}\right) \cdot (3 + 2\sqrt{2})\right]} = 8.76 \text{ x } 10^6 N/m$$

Now with the length given as 4 *meters* and *E* the elastic modulus, found from a table in Chapter 7, to be $200\times 10^9$ $N/m^2$, we find that the cross sectional area is

$$A = 1.76 \text{ x } 10^{-4} m^2 = 176 \text{ mm}^2$$

If the struts were solid and circular, this implies a 15 *mm* diameter.

The stress will be bigger in member *DB* than member *EB*. In fact, from our expression above for $F_D$, we have $\sigma_D = 2 \cdot E \cdot (\Delta_B / L)$ This, evaluated for the particular deflection recorded with one individual supported at *C* and taking *E* as before yields $\sigma_D = 12.6\times 10^6$ $N/m^2$.

If we *idealize* the constitutive behavior as *elastic-perfectly plastic* and take the yield strength as $600\times 10^6$ $N/m^2$, we conclude that we could suspend *forty-seven individuals*, each a hefty weight of 800 *Newtons* before the onset of yield in the strut *BD*, before collapse becomes a possibility.[13] But will it collapse at that point? No, not in this idealized world anyway. Member *EB* has yet to reach its yield strength; once it does, then the structure, again in this idealized world, can support no further increase in end load without infinite deflection and deformations of the struts.

13. This is a strange kind of problem – using the observed displacement under a known load to calculate, to *back-out* the cross sectional areas of the struts. The ordinary, *politically correct* textbook problem would specify the area and everything else you needed (but not a wit more) and ask you to determine the tip displacement. But nowadays machines are given that kind of straight-forward problem to solve. A more challenging kind of dialogue in Engineering Mechanics – common to *diagnostic* situations where your structure is not behaving as expected, when something goes wrong, deflects too much, fractures too soon, resonates at too low a frequency, and the like – demands that you construct different scenarios for the observed behavior, e.g., (did it deflect too much because the top strut exceeded the yield strength?), and test their validity. The fundamental principles remain the same, the language is the same language, but the context is much richer; it places greater emphasis upon your ability to formulate the problem, to construct a story that explains the system's behavior. Often, in these situations, you will not have, or be able to obtain, full or complete information about the structure. In this case, *backing out* the area of the struts might be just one step in diagnosing and explaining the observed and often mystifying, behavior.

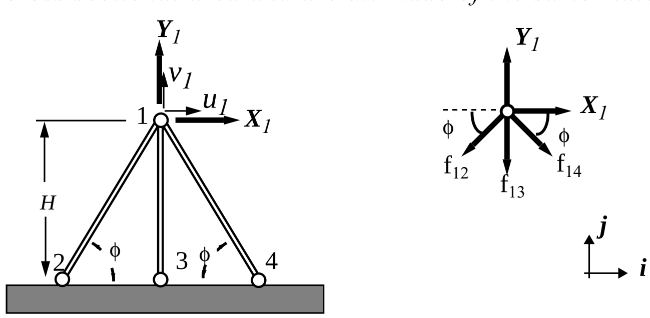
## 5.2 Matrix analysis of Truss Structures - Displacement Formulation

The problems and exercises we have assigned to date have all been amenable to solution *by hand.* We now consider a method of analysis especially well suited for truss structures that takes advantage of modern computer power and allows us to address structures with many nodes and members. Our aim is to find all internal member forces and all nodal displacements given some external forces applied at the nodes. In this, we make use of a *displacement formulation* of the problem; the unknowns of the final set of equations we give the computer to solve are the components of the node displacements. We use matrix notation in our formulation as an efficient and concise way to represent the large number of equations that enter into our analysis.

These equations will account for i) equilibrium of internal member forces and external forces applied at the nodes, ii) the force/deformation behavior of each member and iii) compatibility of the extension and contraction of members with the displacements at the nodes. If the structure is statically determinate and we seek only to determine the forces in the truss members, we need only consider the first of these three sets of equations. If, however, we want to go on and determine the displacements of the nodes as well, we then must consider the full set of relations. If the truss is *statically indeterminate*, if it has redundant members, then we must always, by necessity, consider the deformations of members and displacements of nodes as well as satisfy the equilibrium equations.

To illustrate the *displacement method* we do two examples, one that could be done by hand, the second that is more efficiently done by computer.

***Exercise 5.3*** *– The members of the redundant structure shown below have the same cross sectional area and are all made of the same material. Show*




*that the equations expressing force equilibrium of node #1 in the x and y directions, when phrased in terms of the unknown displacements of the node,* $u_1$, $v_1$, *take the form*

$$2(AE/L)\cdot(\sin\phi\cdot\cos\phi^2)\cdot u_1 = X_1 \qquad and \qquad (AE/L)\cdot(1+2\sin\phi^3)\cdot v_1 = Y_1$$

**Note: We let $X$ and $Y$ designate the $x,y$ components of the applied force at the node, while $u_1$ and $v_1$ designate the corresponding components of displacement of the node.**

**Equilibrium.** with respect to the undeformed configuration, of node #1

$$-f_{12}cos\phi + f_{14}cos\phi + X_1 = 0 \qquad and \qquad -f_{12}sin\phi - f_{13} - f_{14}sin\phi + Y_1 = 0$$

These are two equations in three unknowns as we expected since the structure is redundant. In matrix notation they take the form:

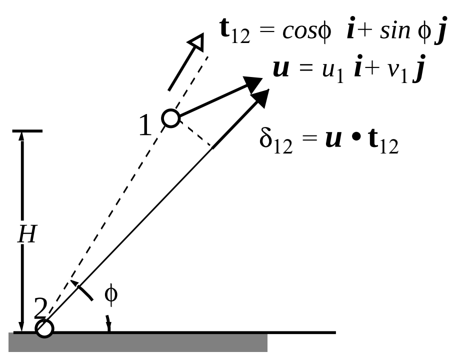
$$\begin{bmatrix} cos\phi & 0 & -cos\phi \\ sin\phi & 1 & sin\phi \end{bmatrix} \begin{bmatrix} f_{12} \\ f_{13} \\ f_{14} \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}$$

**Compatibility of deformation** of the three members is best viewed from the following perspective: Imagine an arbitrary *displacement* of node #1, a vector with two scalar $x,y$ components

$$\boldsymbol{u} = u_1\boldsymbol{i} + v_1\boldsymbol{j}$$

where, as usual, **i, j** are two unit vectors directed along the $x,y$ axes respectively.

We take as a measure of the **member deformation**, say of member 1-2, **the projection of the displacement upon the member**. We must be careful to take account if the member extends or contracts. In the second example we show a way to formally do this bit of accounting. Here we rely upon a sketch. Shown below is member 1-2 and an arbitrary node displacement **u** drawn as if both of its components were positive.




The projection upon the member is given by the scalar, or *dot* product

$$\delta_{12} = \boldsymbol{u} \bullet \boldsymbol{t}_{12} \quad \text{where} \quad \boldsymbol{t}_{12} = cos\phi\boldsymbol{i} + sin\phi\boldsymbol{j}$$

is a unit vector directed as shown, along the member in the direction of a positive extension.

We obtain

$$\delta_{12} = u_1cos\phi + v_1sin\phi$$
$$\delta_{14} = -u_1cos\phi + v_1sin\phi$$
$$\delta_{13} = v_1$$

These three equations relate the three *member deformations* to the two *nodal displacements*[14]. If they are satisfied, we can rest assured that our structure remains all of one piece in the deformed configuration. In matrix notation, they take the form:

$$\begin{bmatrix}\delta_{12}\\ \delta_{13}\\ \delta_{14}\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi\\ 0 & 1\\ -\cos\phi & \sin\phi\end{bmatrix}\begin{bmatrix}u_1\\ v_1\end{bmatrix}$$

Keeping count, we now have five scalar equations for eight unknowns, the three member forces, the three member deformations, and the two nodal displacements. We turn now to the three...

**Force-Deformation** relations are the usual for a truss member, namely

$$f_{12} = (AE/L_{12}) \cdot \delta_{12} \qquad f_{13} = (AE/L_{13}) \cdot \delta_{13} \qquad f_{14} = (AE/L_{14}) \cdot \delta_{14}$$

The lengths may be expressed in terms of $H$, e.g.,

$$L_{12} = L_{14} = H/\sin\phi \qquad and \qquad L_{13} = H$$

These, in matrix notation, take the form

$$\begin{bmatrix}f_{12}\\ f_{13}\\ f_{14}\end{bmatrix} = \begin{bmatrix}\left(\frac{AE\sin\phi}{H}\right) & 0 & 0\\ 0 & \left(\frac{AE}{H}\right) & 0\\ 0 & 0 & \left(\frac{AE\sin\phi}{H}\right)\end{bmatrix}\begin{bmatrix}\delta_{12}\\ \delta_{13}\\ \delta_{14}\end{bmatrix}$$

**Displacement Formulation**. first expressing the member forces in term of the nodal displacements using compatibility and the force-deformation equations, (in matrix notation)

$$\begin{bmatrix}f_{12}\\ f_{13}\\ f_{14}\end{bmatrix} = \begin{bmatrix}\left(\frac{AE\sin\phi}{H}\right) & 0 & 0\\ 0 & \left(\frac{AE}{H}\right) & 0\\ 0 & 0 & \left(\frac{AE\sin\phi}{H}\right)\end{bmatrix}\begin{bmatrix}\cos\phi & \sin\phi\\ 0 & 1\\ -\cos\phi & \sin\phi\end{bmatrix}\begin{bmatrix}u_1\\ v_1\end{bmatrix}$$

then substitute for the forces in the equilibrium equations. We have, again continuing with our matrix representation:

---

14. Note that the horizontal displacement component, $u_1$, engenders no elongation or contraction of the middle, vertical member. This is a consequence of our assumption of small displacements and rotations.

$$\begin{bmatrix} cos\phi & 0 & -cos\phi \\ sin\phi & 1 & sin\phi \end{bmatrix} \cdot \begin{bmatrix} \left(\frac{AE\,sin\phi}{H}\right) & 0 & 0 \\ 0 & \left(\frac{AE}{H}\right) & 0 \\ 0 & 0 & \left(\frac{AE\,sin\phi}{H}\right) \end{bmatrix} \cdot \begin{bmatrix} cos\phi & sin\phi \\ 0 & 1 \\ -cos\phi & sin\phi \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}$$

Carrying out the matrix products yields a set of two scalar equations for the two nodal displacements:

$$\left(\frac{AE}{H}\right)\begin{bmatrix} 2\,sin\phi(cos\phi)^2 & 0 \\ 0 & 1+2\,sin^3\phi \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}$$

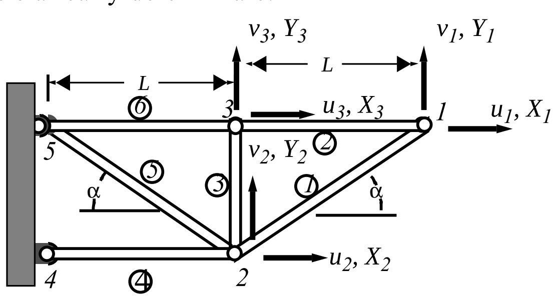
These are the *equilibrium equations in terms of displacements*. They can be easily solved since they are *uncoupled*, that is each can be solved independently for one or the other of the nodal displacements. The symmetry of the structure is the reason for this happy outcome. This becomes clear when we write them out according to our more ordinary habit and obtain what we sought to show:

$$(AE/H)\cdot(2\,sin\phi\,cos\phi^2)u_1 = X_1$$

and

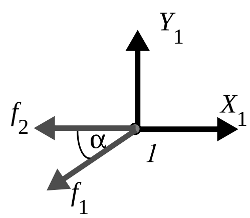
$$(AE/H)\cdot(1+2\,sin\phi^3)v_1 = Y_1$$

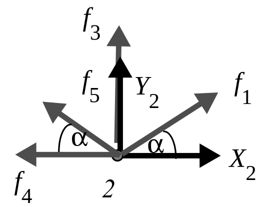
Unfortunately this decoupling doesn't occur often in practice as a second example shows. We turn to that now, a more complex structure, which in a first instance we take as statically determinate.

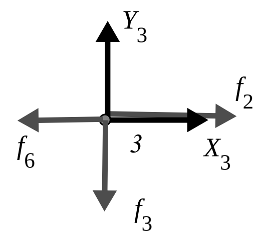



Now consider the truss structure shown above. Although this system is more complex than the previous example in that it has more *degrees of freedom* – six scalar nodal displacements versus two for the simpler truss – the structure is less complex in that it is statically determinate; there are no redundant or unnecessary members; remove any member and the structure would collapse.

We first develop a set of equilibrium equations by isolating each of the free nodes and requiring the sum of all forces, internal and external, to vanish. In this, lower case *f* will represent member forces, assumed to be positive when the member is in tension, and upper case *X* and *Y*, the *x* and *y* components of the externally applied forces.




$- f_2 - f_1\cos\alpha + X_1 = 0$ $\quad - f_4 - f_5\cos\alpha + f_1\cos\alpha + X_2 = 0$ $\quad - f_6 + f_2 + X_3 = 0$

$- f_1\sin\alpha + Y_1 = 0$ $\quad f_5\sin\alpha + f_3 + f_1\sin\alpha + Y_2 = 0$ $\quad - f_3 + Y_3 = 0$

These six equations for the six unknown member forces can be put into matrix form

$$\begin{bmatrix} \cos\alpha & 1 & 0 & 0 & 0 & 0 \\ \sin\alpha & 0 & 0 & 0 & 0 & 0 \\ -\cos\alpha & 0 & 0 & 1 & \cos\alpha & 0 \\ -\sin\alpha & 0 & -1 & 0 & -\sin\alpha & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ X_3 \\ Y_3 \end{bmatrix}$$

We could, if we wish at this point, solve this system of six linear equations for the six unknown member forces *f*. If you are so inclined you can apply the methods you learned in your mathematics courses about existence of solutions, about solving linear systems of algebraic equations, and verify for yourself that indeed a *unique* solution does exist. And given enough of your spare time, I wager you could actually carry through the algebraic manipulations and obtain the solution. But our purpose is not to burden you with ordinary menial exercise but rather to show you how to formulate the problem for computer solution. We will let it do the menial and mundane work.

Something is lost, something is gained when we turn to the machine to help solve our problems. The expressions you would obtain by hand for the internal forces would be explicit functions of the applied forces and the parameter $\alpha$. For example, the second equation alone gives

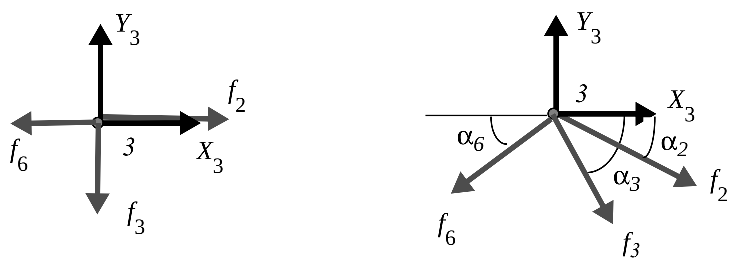
$$f_1 = Y_1/(\sin\alpha)$$

The computer, on the other hand, would produce, using the kinds of software common in industry, a solution for specific *numerical* values of the member forces if provided with a specific, numerical value for $\alpha$ and specific numerical values for the externally applied forces at the nodes as input. Of course the computer does this very fast, compared to the time it would take you to produce a solution by hand. And, if need be, with the machine you can make many runs and discover how your results vary with $\alpha$.

But note: How the solution changes with changes in the external forces applied at the nodes is a simpler matter: since the solutions will be *linear* functions of the *X* and *Y*'s you can scale your results for one loading condition to get another loading condition. That's what *linear systems* means.

**A small detour**:

The system is linear because we assumed that the structure experiences only *small displacements and rotations*. We wrote our equilibrium equations with respect to the *undeformed geometry* of the structure. If we thought of the structure otherwise, say as made of rubber and allowed for large displacements, our free-body diagrams would be incorrect as they stand above. For example, the situation at node 3 would appear as at the right rather than as before (at the left)




$$-f_6 + f_2 + X_3 = 0 \qquad -f_6\cos\alpha_6(u) + f_2\cos\alpha_2(u) + X_3 = 0$$

$$-f_3 + Y_3 = 0 \qquad -f_3\sin\alpha_3(u) - f_6\sin\alpha_6(u) - f_2\sin\alpha_2(u) + Y_3 = 0$$

and our equilibrium equations would now have the more complex form shown.

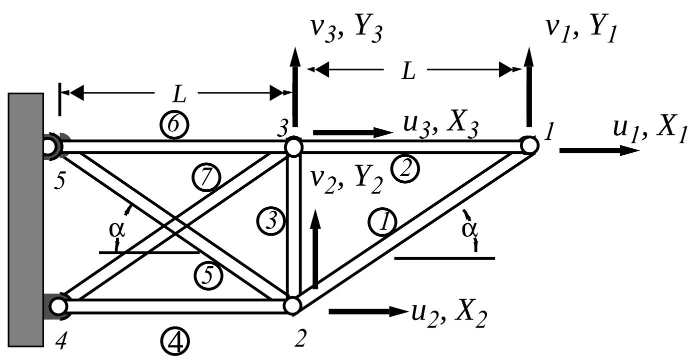
In these, the alpha's will be unknown functions of all the nodal displacements, for example $\alpha_2$ will depend upon the displacement of node 3 relative to node 1. We say that the equilibrium equations depend upon the displacements.

But the displacements are functions of the extensions and contractions of the members. These, in turn, are functions of the forces in the members which means that the equations of equilibrium are no longer *linear*. The entries in our square matrix, the coefficients of the unknown forces in our system of six equilibrium equations, depend upon the member forces themselves.

Fortunately, although we did so in our introductory exercise in Chapter 1, you will not be asked to consider large deformations and rotations. The reason is that most structures *do not* experience large deflections and rotations. If they do they are probably in the process of disintegration and failure. Indeed, eventually we will entertain a discussion of *buckling* which ordinarily, though not always, is a mode of failure. We leave, then, the study of such complex, but interesting, modes of behavior to other scholars. Our detour is complete; we return now to more ordinary behavior.

Out in the so-called *real* world, where truss structures span canyons, support aerospace systems, and have hundreds of nodes and members, complexity requires the use of the computer. Imagine a *three-dimensional* truss with 100 nodes. Our linear system of equilibrium equations would number 300; we say that the system has 300 *degrees of freedom*. That is, 300 displacement components are required to fully specify the deformed configuration of the structure. But that is not the end of it: if the structure includes redundant members and hence is statically indeterminate, other equations which relate the member forces to member deformations and still others relating member deformations to node displacements must be written down and solved together with the equilibrium equations. You could still, theoretically, solve all of these hundreds of equations by hand but if you want to remain industrially competitive, if you want to win the bid, you will need the services of a computer.

To illustrate how our system is complicated by adding a redundant member, we connect nodes 3 and 4 with an additional member, number 7 in the figure.




The number of linearly independent equilibrium equations remains the same, namely six, but two of the equations, expressing horizontal and vertical equilibrium of forces at node 3, now include the additional unknown member force $f_7$. Leaving to you the task of amending the free-body diagram of node #3, we have

$$- f_6 + f_2 - f_7 cos\alpha + X_3 = 0$$
$$- f_3 - f_7 sin\alpha + Y_3 = 0$$

With six equations for seven unknowns our problem becomes *statically indeterminate* or *equilibrium indeterminate* as some would prefer. The difficulty is not in finding a solution; indeed, there are an *infinity* of possible solutions. For example we could choose the force in member six to be equal to zero and then solve for all the other member forces. Or we could choose it to equal $X_2$ and solve, or 10 *lbs*, or 2000 *newtons*, or 2.3 *elephants*, (just be careful with your units), whatever. Once having arbitrarily specified the force in member six, or the force in any single member for that matter, the six equations will yield values for the forces in all the remaining six members. The difficulty is not in finding a solution, it is in finding a *unique* solution. The problem is indeterminate.

This unique solution, whatever it is, is going to depend upon the kind of member we add to the structure as member number seven. It will depend upon the material properties and cross-sectional area of this new member; for that matter, it will depend upon the force/deformation behavior of *all* members. If the first six members are made of steel and have a cross-sectional area of ten square inches and member seven is a rubber band, we would not expect much difference in our solution for the forces in the steel members when compared to our original solution for those member forces without member seven. If, on the other hand, the added member is also made of steel and has a comparable cross-sectional area, all bets are off, or rather on. The effect of the new member will be significant; the member forces will be substantially different when compared to the statically determinate solution.

Our strategy for solving the statically indeterminate problem is the same one we followed in the previous exercise: We will express all seven unknown internal forces *f* in terms of the seven, unknown, member deformations which we will designate by δ. We will then develop a method for expressing the member deformations, the δ's, in terms of the *x* and *y* components of nodal displacements *u* and *v*. There are six of these latter unknowns. After substitution, we will then obtain our six equilibrium equations in terms of the *six* unknown displacement components. *Voila*, a *displacement formulation*.

***Equilibrium***

The full set of six equilibrium equations in terms of the seven unknown member forces may be written in matrix form as

$$\begin{bmatrix} \cos\alpha & 1 & 0 & 0 & 0 & 0 & 0 \\ \sin\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ -\cos\alpha & 0 & 0 & 1 & \cos\alpha & 0 & 0 \\ -\sin\alpha & 0 & -1 & 0 & -\sin\alpha & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & \cos\alpha \\ 0 & 0 & 1 & 0 & 0 & 0 & \sin\alpha \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ X_3 \\ Y_3 \end{bmatrix}$$

or in a condensed form as: $[A]\{\boldsymbol{f}\} = \{\boldsymbol{X}\}$

Note that the array [$A$] has six rows and seven columns; there are but six equations for the seven unknown internal forces.

### *Force-Deformation*

We assume that the truss members behave like linear springs and, as before, take the member force generated in deformation of the structure as proportional to their change in length $\delta$. We introduce the symbol $k$ for the expression ($AE/L$) where $A$ is the member cross-sectional area, $L$ its length, and $E$ its modulus of elasticity. For example, for member number 1, we take

$$f_1 = k_1 \cdot \delta_1 \quad \text{where} \quad k_1 = A_1 E_1 / L_1$$

In matrix form,

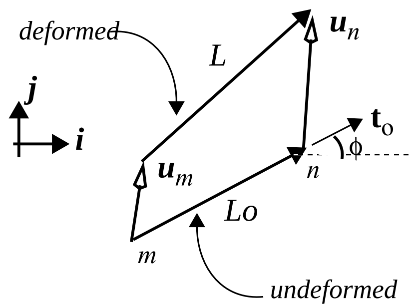
$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix} = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_7 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \end{bmatrix}$$

or again, in condensed form:

$$\left[f\right] = \left[k\right] \cdot \left[\delta\right]$$

### *Compatibility of Deformation*

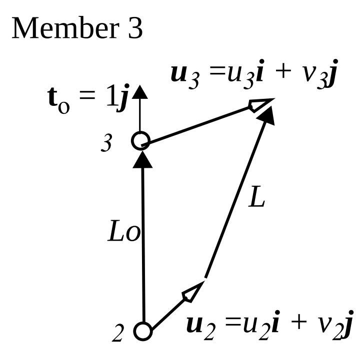
Taking stock at this point we see we have *thirteen* equations but *fourteen* unknowns; the latter include seven member forces $f$ and seven member deformations $\delta$. In this our final step, we introduce another six unknowns, namely the $x$ and $y$ components of the displacements at the nodes and require that the member deformations be consistent with these displacements. *Seven* equations, one for each member, are required to ensure compatibility of deformation. This will bring our totals to *twenty* equations for *twenty* unknowns and allow us to claim victory.




To relate the δ's to the node displacements we consider an arbitrarily oriented member in its undeformed position, then in its deformed state, a state defined by the displacements of its two end nodes. In the following derivation, bold face type will indicate a vector quantity.

Consider a member with end nodes numbered *m* and *n*. Let $\boldsymbol{u}_m$ be the vector displacement of node *m*. In terms of its *x* and *y* scalar components we have:

$$\boldsymbol{u}_m = u_m \boldsymbol{i} + v_m \boldsymbol{j}$$




where ***i*** and ***j*** are unit vectors in the *x,y* directions. A similar expression may be written for $\boldsymbol{u}_n$.

Let $\boldsymbol{L}_0$ be a vector which lies along the member, going from *m* to *n*, in its original, undeformed state and ***L*** a vector along the member in its displaced, deformed state. Vector addition allows us to write: $\boldsymbol{L_o} + \boldsymbol{u_n} = \boldsymbol{u_m} + \boldsymbol{L}$

Now consider the projection of all of these vector quantities upon a line lying along the member in its original, undeformed state, that is along $\boldsymbol{L}_0$. Let $\boldsymbol{t}_0$ be a unit vector in that direction, directed from *m* to *n*.

$$\boldsymbol{t}_o = \cos\phi \boldsymbol{i} + \sin\phi \boldsymbol{j}$$

The projection of $\boldsymbol{L}_0$ upon itself is just the original length of the member, the magnitude of $\boldsymbol{L}_0$, $L_0$. The projections of the node displacements are given by the scalar products $\boldsymbol{t}_0 \bullet \boldsymbol{u}_m$ and $\boldsymbol{t}_0 \bullet \boldsymbol{u}_n$. Similarly the projection of ***L*** is $\boldsymbol{t}_0 \bullet \boldsymbol{L}$ which we take as approximately equal to the magnitude of ***L***. This is a crucial step. It is only legitimate if the member experiences *small rotations*. But note, this is precisely the assumption we made in writing out our equilibrium equations.

Our vector relationship then yields, after projection upon the direction $\boldsymbol{t}_0$ of all of its constituents

$$L - L_o \approx \boldsymbol{u}_n \cdot \boldsymbol{t}_o - \boldsymbol{u}_m \cdot \boldsymbol{t}_o$$

or since the difference of the two lengths is the member's extension, we have

$$\delta = \boldsymbol{u}_n \cdot \boldsymbol{t}_o - \boldsymbol{u}_m \cdot \boldsymbol{t}_o$$

For member 1, for example, carrying out the scalar products we have

$$\delta_3 = v_3 - v_2$$

Note how the horizontal displacement components, the u components, do not enter into this expression for the extension (or compression if $v_2 > v_3$) of member 3. That is, any displacement perpendicular to the member does not contribute to its change its length! This is clearly only approximately true, only true for *small displacements and rotations.*

Similar equations can be written for each member in turn. In some cases, $\phi$ is zero, in other cases a right angle. The full set of seven compatibility relationships, one for each member, can be written in matrix form as:

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \end{bmatrix} = \begin{bmatrix} cos\alpha & sin\alpha & -cos\alpha & -sin\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & cos\alpha & -sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & cos\alpha & sin\alpha \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

In condensed form we write $\left[\delta\right] = \left[A\right]^T \cdot \left[u\right]$

where $[A]^T$ is the *transpose* of the matrix appearing in the equilibrium equations [*A*]. The consequence of this seemingly happenstance event will be come clear in the final result.

### *Equilibrium in terms of Displacement*

We now do some substitution to obtain the equilibrium equations in terms of the displacement components at the nodes, all the *u*'s and *v*'s. We first substitute for the member forces *f*, their representation in terms of the member deformations $\delta$ and obtain:

$$\left[A\right] \cdot \left[k\right] \cdot \left[\delta\right] = \left[X\right]$$

Now substituting for the $\delta$ column matrix its representation in terms of the node displacements we obtain:

$$\left[A\right] \cdot \left[k\right] \cdot \left[A\right]^T \cdot \left[u\right] = \left[X\right]$$

which are the *six* equilibrium equations with the *six* displacement components as unknowns.

The matrix product $[A][k][A]^T$ can be carried out in more spare time. We designate the result by [*K*] and call it the *system stiffness matrix*. It and all of its elements are shown below: In this, *c* is shorthand for *cos*$\alpha$ and *s* shorthand for *sin*$\alpha$.

$$\left[K\right] = \left[A\right] \cdot \left[k\right] \cdot \left[A\right]^T$$

$$[K] = \begin{bmatrix} (k_1c^2+k_1) & k_1cs & -k_1c^2 & -k_1cs & k_2 & 0 \\ k_1cs & (k_1s^{2)} & -k_1cs & -k_1s^2 & 0 & 0 \\ -k_1c^2 & -k_1cs & \left(k_4+k_1c^2+k_5c^2\right) & (k_1cs-k_5cs) & 0 & 0 \\ -k_1cs & -k_1s^2 & (k_1cs-k_5cs) & (k_3+k_1s^2+k_5s^2) & 0 & -k_3 \\ k_2 & 0 & 0 & 0 & (k_2+k_6+k_7c^2) & k_7cs \\ 0 & 0 & 0 & -k_3 & k_7cs & (k_3+k_7s^2) \end{bmatrix}$$

$[K]$ is *symmetric* (and will always be!) and, for our example is *six by six*.

The equilibrium equations in terms of displacement are, in condensed form

$$\boxed{[K]\cdot[u] = [X]}$$

This is the set of equations the computer solves given adequate numerical values for

- the material properties including the *Young's modulus* or modulus of elasticity, *E*, and the member's cross-sectional area *A*,
- member nodes and their coordinates, from which member lengths may be figured, and subsequently together with the material properties, the member stuffiness, (AE/L), computed,
- the externally applied forces at the nodes,
- specification of any fixed degrees of freedom, i.e., which nodes are pinned.

The computer, in effect, *inverts* the system, or global, stiffness matrix $[K]$, and computes the node displacements $u$ and $v$ given values for the applied forces $X$ and $Y$. Once the displacements have been found, the deformations can be computed from the compatibility relations. Making use of the force/deformation relations in turn, the deformations yield values for the member forces. All then has been resolved, the solution is complete.

Before ending this section, one final observation. A useful physical interpretation of the elements of the system stiffness matrix is available: In fact, the elements of any column of the $[K]$ matrix can be read as the external forces that are required to produce or sustain a special state of deformation, or system of node displacements – namely a unit displacement corresponding to the chosen column and zero displacements in all other degrees of freedom. This interpretation follows from the rules of matrix multiplication.

### A Note on Scaling

It is useful to consider how the solution for one particular structure of a specified geometry and subject to a specific loading can be applied to another structure of similar geometry and similar loading. By "similar loading" we mean a load vector which is a scalar multiple of the other. By "similar geometry" we mean a structure whose member lengths are a scalar multiple of the corresponding member lengths of the other - in which case all angles are preserved.

For similar loading, relative to some reference solution designated by a superscript "*", i.e.,

$$[K] \cdot [u^*] = [X^*]$$

we have, if [X] = β [X*] simply that the displacement vector scales accordingly, that is, from

$$[K] \cdot [u] = [X] = \beta \cdot [X^*]$$

we obtain [u] = β [u*] .

This is a consequence of the *linear* nature of our system (which, in turn, is a consequence of our assumption of relatively small displacements and rotations). What it says is that if you have solved the problem for one particular loading, then the solution for an infinity of problems is obtained by scaling your result for the displacements (and for the member forces as well) by the factor β which can take on an infinity of values.

For similar geometries, we need to do a bit more work. We note first that for both statically determinate and indeterminate systems, the only way length enters into our analysis is through the member stiffness, k, where $k_j = AE/L_j$ . (We assume for the moment that the cross-sectional areas and the elastic modulae are the same for each member). The entries in the matrix [A], and so $[A^T]$ are only functions of the angles the members make, one with another.

Let us designate some reference geometry, drawn in accord with some reference length scale, by a superscript "*", a reference structure in which the member lengths are defined for all members, j= 1,n by

$$L^*_j = \beta_j \cdot L^*$$

The force-deformation relations $[f] = [k] \cdot [\delta]$ can then be written

$$[f] = (1/L^*) \cdot [k_\beta] \cdot [\delta]$$

where the elements of the matrix $[k_\beta]$ are given by $AE/\beta_j$ .

Re-doing our derivation of the equilibrium equations expressed in terms of displacements yields.

$$\left[K_\beta\right] \cdot \left[u^*\right] = L^* \cdot \left[X\right]$$

where the stiffness matrix $\left[K_\beta\right]$ is given by

$$\left[K_\beta\right] = \left[A\right] \cdot \left[k_\beta\right] \cdot \left[A\right]^T$$

Note that this is only dependent upon the relative lengths of the members, upon the $\beta_j$. Then if we change length scales, say our reference length becomes L, we have to solve

$$\left[K_\beta\right] \cdot \left[u\right] = L \cdot \left[X\right]$$

But the solution to this is the same, in form, as the solution to the "*" problem, differing only by the scale factor L/L*. Hence, solving the reference problem gives us the solution for an infinite number of geometrically similar structures bearing the same loading. (Note that if the loading is scaled down by the same factor by which the geometry is scaled up, the solution does not change!)

## 5.3 Energy Methods [15]

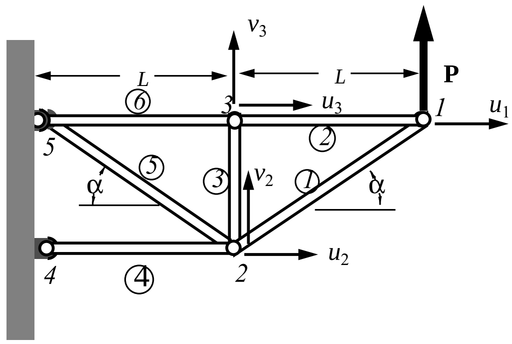
We have now all the machinery, concepts and principles, we need to solve any truss problem. The structure can be equilibrium indeterminate or determinate. It matters little. The computer enables the treatment of structures with many degrees of freedom, determinate and indeterminate.

But before the computer existed, mechanicians solved truss structure problems. One of the ways they did so was via methods rooted in an alternative perspective - one which builds on the notions of work and energy. We develop some of these methods in this section but will do so based on the concepts and principles we are already familiar with, without reference to energy.

The first method may be used to determine the *displacements of a statically determinate truss structure*. Generalization to indeterminate structures will follow.

---

15. The perspective adopted here retains some resemblance to that found in Strang, G., *Introduction to Applied Mathematics*, Wellesley-Cambridge Press, Wellesley, MA., 1986. I use "Energy Methods" only as a label to indicate what this section is meant to replace in other textbooks on Statics and Strength of Materials.




Before proceeding, we review how we might determine the displacements following the path taken in developing the stiffness matrix. We take as an example the statically determinate example of the last section. We simplify the system, applying but one load, **P** in the vertical direction at node 1.

The system is determinate so we solve for the six member forces using the six equations of equilibrium obtained by isolating the structure's three free nodes.

$$- f_2 - f_1\cos\alpha = 0 \qquad - f_4 - f_5\cos\alpha + f_1\cos\alpha = 0 \qquad - f_6 + f_2 = 0$$

$$- f_1\sin\alpha + P = 0 \qquad f_5\sin\alpha + f_3 + f_1\sin\alpha = 0 \qquad - f_3 = 0$$

These give:

$f_1 = P/\sin\alpha$; $f_2 = -P\cos\alpha/\sin\alpha$; $f_3 = 0$; $f_4 = 2P\cos\alpha/\sin\alpha$;
$f_5 = -P/\sin\alpha$; and $f_6 = -P\cos\alpha/\sin\alpha$

where a positive quantity means the member is in tension, a negative sign indicates compression.

With proceed to determine member deformations, $[\delta]$, from the force/deformation relationships

$$[\delta] = [\,k_{diag}\,]^{-1}\,[f]$$

that is, from $\delta_1 = f_1/k_1$, $\delta_2 = f_2/k_2$, ... etc; where the k's are the individual member stiffness, e.g.,

$$k_1 = A_1E_1/L_1 \quad \text{... etc.}$$

Then, from the compatibility equation relating the six member deformations to the six displacement components at the nodes,

$$[\delta] = [\,A\,]^T[u]$$

we solve this system of six equations for the six displacement components $u_1$, $v_1$, $u_2$, $v_2$, $u_3$, $v_3$. That's it.

### A Virtual Force Method

Now consider the alternative method:

We start with the compatibility condition: $[\delta] = [A]^T[u]$

and take a totally unmotivated step, multiplying both sides of this equation by the transpose of a column vector whose elements may be anything whatsoever;

$$[f^*]^T[\delta] = [f^*]^T[A]^T[u]$$

This arbitrary vector bears an asterisk to distinguish from the vector of member forces acting in the structure.

At this point, the elements of [f*] could be any numbers we wish, e.g., the price of coffee in the six largest cities of the US (it has to have six elements because the expressions on both sides of the compatibility equation are 6 by 1 matrices). But now we manipulate this relationship, taking the transpose of both sides and write

$$[\delta]^T[f^*] = [u]^T[A][f^*]$$

then consider the vector [f*] to be *a vector of member forces*, ***any*** set of member forces that satisfies the equilibrium requirements for the structure, i.e.,

$$[A][f^*] = [X^*]$$

So [X*] is arbitrary, because [f*] is quite arbitrary - we can envision many different vectors of applied loads.

With this, our compatibility pre-multiplied by our arbitrary vector, now read as member forces, becomes

$$[\delta]^T[f^*] = [u]^T [X^*] \qquad \text{or} \qquad [u]^T [X^*] = [\delta]^T[f^*]$$

(Note: The dimensions of the quantity on the left hand side of this last equation are displacement times force, or work. The dimensions of the product on the right hand side must be the same).

Now we choose [X*] in a special way; we take it to be a unit load, a *virtual force*, along a single degree of freedom, all other loads zero. For example, we take

$$[X^*]^T = [\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ ]$$

a unit load in the vertical direction at node 3 in the direction of $v_3$.

Carrying out the product $[u]^T$ [X*] in the equation above, we obtain just the displacement component associated with the same degree of freedom, $v_3$ i.e.,

$$v_3 = [\delta]^T[f^*]$$

We can put this last equation in terms of member forces (and member stiffness) alone using the force/deformation relationship and write:

$$v_3 = [f]^T[k]^{-1}[f^*]$$

And that is our special method for determining displacements of a statically determinate truss. It requires, first, solving equilibrium for the “actual” member forces given the “actual” applied loads. We then solve *another* force equilibrium problem - one in which we apply a unit load at the node we seek to determine a displacement component and in the direction of that displacement component.

With the "starred" member forces determined from equilibrium, we carry out the matrix multiplication of the last equation and there we have it.

We emphasize the difference between the two member force vectors appearing in this equation; [f] in plain font, is the vector of the actual forces in structure given the actual applied loads. [f*] with the asterisk, on the other hand, is some, originally arbitrary, force vector which satisfies equilibrium — an equilibrium solution for member forces corresponding to a **unit loading** in the vertical direction at node 3.

Continuing with our specific example, the virtual member forces corresponding to the unit load at node 3 in the vertical direction are, from equilibrium:

$f^*_1 = 0$

$f^*_2 = 0$

$f^*_3 = 1$

$f^*_4 = \cos\alpha / \sin\alpha$

$f^*_5 = -1/\sin\alpha$

$f^*_6 = 0$

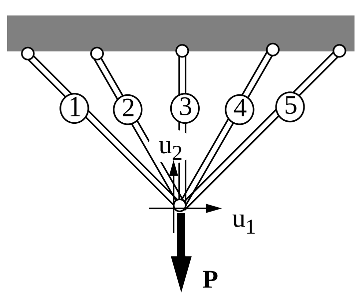
We these, and our previous solution for the *actual* member forces, we find

$$v_3 = (P/k_4)(2\cos\alpha/\sin\alpha)(\cos\alpha/\sin\alpha) + (P/k_5)/\sin^2\alpha$$

If the members all have the same cross sectional area and are made of the same material, then the ratio of the member stiffness goes inversely as the lengths so $k_5 = \cos\alpha \; k_4$

and, while some further simplification is possible, we stop here.

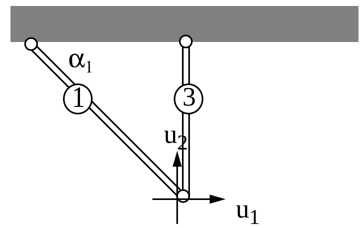
## Virtual Force Method for Redundant Trusses - Maxwell/Mohr Method.




Let's say we have a redundant structure as shown at the left. Now assume we have found all the actual forces, $f_1, f_2, \ldots f_5$, in the members by an alternative method yet to be disclosed (it immediately follows this preliminary remark). The actual loading consists of force components $X_1$ and $X_2$ applied at the one free node in directions indicated by $u_1$ and $u_2$.

Now say we want to determine the horizontal component of displacement, $u_1$; Proceeding in accord with our Force Method #1, we must find an equilibrium set of member forces given a unit load applied at the free node in the horizontal direction.

Since the system is redundant, our equilibrium equations number 2 but we have 5 unknowns. The system is indeterminate: it does not admit of a unique solution. It's not that we can't find a solution; the problem is we can find *too many* solutions. Now since our "starred" set of member forces need only satisfy equilibrium, we can arbitrarily set the redundant member forces to zero, or, in effect, *remove them from the structure.* The figure at the right shows one possible choice




*For a unit force in the horizontal direction, we have*

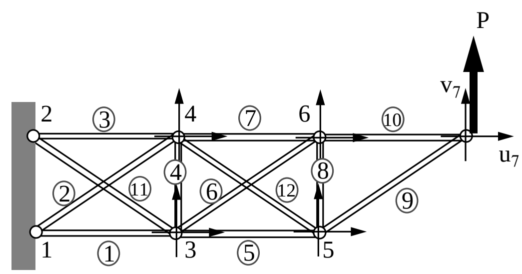
$f_1^* = 1/\cos\alpha_1$ and $f_3^* = -1 \sin\alpha_1 / \cos\alpha_1$

so the displacement in the horizontal direction, assuming again we have determined the actual member forces, is

$u_1 = (f_1/k_1)(1/\cos\alpha_1) - (f_3/k_3)(1 \sin\alpha_1 / \cos\alpha_1)$

(Note: If the structure is symmetric in member stiffness, k, then this component of displacement, for a vertical load alone, should vanish. This then gives a relationship between the two member forces).

We now develop an alternative method to determine the actual member forces in statically indeterminate truss structures. Consider, for example, the redundant structure shown at the right. We take members 11 and 12 as redundant and write equilibrium in a way that explicitly distinguishes the forces in these two redundant members from the forces in all the other members. The reasons for this will become clear as we move along.




$$\left[\begin{array}{c|c} A_d & A_r \end{array}\right] \cdot \left[\begin{array}{c} f_d \\ \hline f_r \end{array}\right] = [X]$$

In this, because there are 5 unrestrained nodes, each with two degrees of freedom, the column matrix of external forces, [X], is 10 by 1. Because there are two redundant members, the column matrix $[f_r]$ is 2 by 1. The column matrix of what we take to be "determinate member forces" $[f_d]$ is 10 by 1, i.e., there are a total of

12 member forces. Here, then, are 10 equations for 12 unknowns - an indeterminate system.

The matrix $[A_d]$ has 10 rows and 10 columns and contains the coefficients of the 10 $[f_d]$. The matrix $[A_r]$, containing coefficients of the 2 $[f_r]$, has 10 rows and 2 columns.

Equilibrium can then be re-written

$[A_d][f_d] + [A_r][f_r] = [X]$ or $[A_d][f_d] = - [A_r][f_r] + [X]$

but leave this aside, for now, and turn to compatibility. What we are after is a way to determine the forces in the redundant members without having to explicitly consider compatibility of deformation. Yet of course compatibility must be satisfied, so we turn there now.

The relationship between member deformations and nodal displacements can also be written to explicitly distinguish between the deformations of the "determinant" members and those of the redundant members, that is, the matrix equation $[\delta] = [A]^T[u]$ can be written:

$$\begin{bmatrix} \delta_d \\ \hline \delta_r \end{bmatrix} = \begin{bmatrix} A_d^T \\ \hline A_r^T \end{bmatrix} \cdot \begin{bmatrix} u \end{bmatrix} \quad \text{or} \quad \begin{matrix} [\delta_d] = [A_d]^T \cdot [u] \\ \text{and} \\ [\delta_r] = [A_r]^T \cdot [u] \end{matrix}$$

The top equation on the right is the one we will work with. As in force method #1, we premultiply by the transpose of a column vector (10 by 1) whose elements can be any numbers we wish. In fact, we multiply by the transpose of a general matrix of dimensions 10 rows and *2 columns* - the 2 corresponding to the number of redundant member forces. The reasons for this will become clear soon enough. We again indicate the arbitrariness of the elements of this matrix with an asterisk. We write

$[f_d^*]^T [\delta_d] = [f_d^*]^T [A_d]^T[u]$

In this $[f_d^*]^T$ is 2 rows by 10 columns and $[\delta_d]$ is 10 by 1.

Now take the transpose and obtain

$[\delta_d]^T[f_d^*] = [u]^T [A_d][f_d^*]$

At this point we choose the matrix $[f_d^*]$ to be very special; each of the two columns of this matrix (of 10 rows) we take to be a solution to equilibrium. The first column is the solution when

- the external forces [X] are all zero and
- the redundant force in member 11 is taken as a *virtual force* of unity.

The second column is the solution for the determinate member forces when

- the external forces [X] are all zero and
- the redundant force in member 12 is taken as a *virtual force* of unity.

That is, from equilibrium,

$[A_d]\,[f_d{*}] = -\,[A_r][\,I\,]$   where [ I ] is the identity matrix.

With this, our compatibility condition becomes

$[\delta_d]^T[f_d{*}] = -\,[u]^T\,[A_r]$   and taking the transpose of this, noting that $[\delta_r] = [A_r]^T[u]$

we have

$$[f_d{*}]^T\,[\delta_d] = -\,[\delta_r]$$

which gives us the redundant member deformations, $[\delta_r]$, in terms of the "determinate" member deformations, $[\delta_d]$.

But we want the member forces too so we now introduce the member force deformations relations which are simple enough, that is

$[\delta_r] = [k_r]^{-1}\,[f_r]$   and   $[\delta_d] = [k_d]^{-1}\,[f_d]$   which enables us to write

$$[f_r] = -\,[k_r][f_d{*}]^T\,[k_d]^{-1}\,[f_d]$$

which, if given the determinate member forces, allows us to compute the redundant member forces.

Substituting, then, back into the equilibrium equations, we can eliminate the redundant forces, expressing the redundant forces in terms of the 10 other member forces, and obtain a system of 10 equations for the 10 unknowns $[f_d]$, namely

$$[\,[A_d] - [A_r][k_r][f_d{*}]^T\,[k_d]^{-1}\,][f_d] = [X]$$

There we have it; a way to determine the member forces in a equilibrium indeterminate truss structure and we don't have to explicitly consider compatibility. What we must do is solve equilibrium several times over; two times to obtain the elements of the matrix $[f_d{*}]$ in accord with the bulleted conditions stated previously, then, finally, the last equation above, given the applied forces [X].

To go on to determine displacements, we can apply force method #1 - apply a unit load according to the displacement component we wish to determine; use the above two equations to determine all **member forces** (with an asterisk to distinguish them from the actual member forces); then, with the artificial, equilibrium satisfying, "starred" member forces, carry out the required matrix multiplications.

We might wonder how we can get away without explicitly considering compatibility on our way to determining the member forces in an indeterminate truss structure. That we did include compatibility is clear - that's where we started. How does it disappear, then, from view?

The answer is found in one special, mysterious feature of our truss analysis. We have observed, but not proven, that the matrix relating displacements to deformations is the transpose of the matrix relating the applied forces to member forces.

That is, equilibrium gives [A][f] = [X]

While, compatibility gives $[\delta] = [A]^T[u]$

Now that is bizarre! A totally unexpected result since equilibrium and compatibility are quite independent considerations. (It's the force/deformation relations that tie the quantities of these two domains together). It is this feature which enables us to avoid explicitly considering compatibility in solving an indeterminate problem. Where does it come from? How can we be sure these methods will work for other structural systems?

## Symmetry of the Stiffness Matrix - Maxwell Reciprocity

The answer lies in that other domain; that of work and energy. In fact, one can prove that if the work done is to be path independent (which defines an elastic system) then this happy circumstance will prevail.

Consider some quite general truss structure, loaded in the following *two* ways: Let the original, unloaded, state of the system be designated by the subscript "*o*".

A first method of loading will take the structure to a state "*a*", where the applied nodal forces $[X_a]$ engender a set of nodal displacements $[u_a]$, then on to state "*c*" where an *additional* applied set of forces $[X_b]$ engender a set of *additional* nodal displacements, $[u_b]$. Symbolically: $o \to a \to c = a + b$ and the work done in following this path may be expressed as[16]

$$Work_{o \to c} = \int_o^c [X]^T \cdot [du] = \int_o^a [X]^T \cdot [du] + \int_a^c [X]^T \cdot [du]$$

and, in that the second integral can be expressed as

$$\int_a^c [X]^T \cdot [du] = \int_a^c [X_a + (X - X_a)]^T \cdot [du] = [X_a]^T \cdot \int_o^b [du] + \int_o^b [X]^T \cdot [du]$$

we have, for this path from o to c:

$$Work_{o \to c} = \int_o^a [X]^T \cdot [du] + \int_o^b [X]^T \cdot [du] + [X_a]^T \cdot [u_b]$$

A second method of loading will take the structure first to state "b", where the applied nodal forces $[X_b]$ engender a set of nodal displacements $[u_b]$, then on to state "*c*" where an *additional* applied set of forces $[X_a]$ engender a set of *addi-*

16. We assume linear behavior as embodied in the stiffness matrix relationship [X] = [K][u].

*tional* nodal displacements, $[u_a]$. Symbolically: $o \rightarrow b \rightarrow c = a + b$ And following the same method, we obtain for the work done:

$$Work_{o \rightarrow c} = \int_o^a [X]^T \cdot [du] + \int_o^b [X]^T \cdot [du] + [X_b]^T \cdot [u_a]$$

Comparing the two boxed equations, we see that for the work done to be path independent we must have

$$[X_a]^T \cdot [u_b] = [X_b]^T \cdot [u_a]$$

or, with [X] = [K][u]

$$[u_a]^T \cdot [K] \cdot [u_b] = [u_b]^T \cdot [K] \cdot [u_a]$$

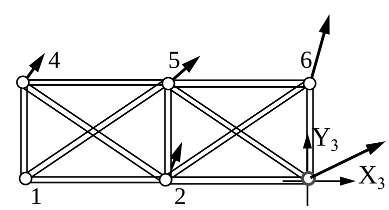
from which we conclude that [K], the stiffness matrix, must be symmetric.

Now, since, as derived in a previous section, $[K] = [A] \cdot [k_{diag}] \cdot [A]^T$, we see how this must be if work done is to be path independent.

## A Virtual Displacement Method.

Given the successful use of equilibrium conditions alone for, not just member forces, but nodal displacements and for indeterminate as well as determinate truss structures, we might ask if we can do something similar using compatibility conditions alone. Here life gets a bit more unrealistic in the sense that the initial problem we pose, drawing on force method #1 as a guide, is not frequently encountered in practice. But it is a conceivable problem - a problem of prescribed displacements. It might help to think of yourself being set down in a foreign culture, a different world, where mechanicians have only reluctantly accepted the reality of forces but are well schooled in displacements, velocities and the science of anything that moves, however minutely.

That is, we consider a truss structure, all of whose displacement components are prescribed, and we are asked to determine the external forces required to give this system of displacements. In the figure at the right, the vectors shown are meant to be the known prescribed displacements. (Node #1 has zero displacement). The task is to find the external forces, e.g., $X_3$, $Y_3$, which will produce this deformed



state and be in equilibrium - and we want to do this without considering equilibrium explicitly!

We start with equilibrium[17]:

$[X] = [A]\,[f]$

and take a totally unmotivated step, multiplying both sides of this equation by the transpose of a column vector whose elements may be anything whatsoever;

$[u^*]^T[X] = [u^*]^T[A]\,[f]$

This arbitrary vector bears an asterisk to distinguish it from the vector of actual displacement prescribed at the nodes.

At this point, the elements of [u] could be any numbers we wish, e.g., the price of coffee in the 12 largest cities of the US (it has to have twelve elements because the expressions on both sides of the equilibrium equation are 12 by 1 matrices). But now we manipulate this relationship, taking the transpose of both sides and write

$[X]^T[u^*] = [f]^T[A]^T[u^*]$

then consider the vector [u*] to be *a vector of nodal displacements*, ***any*** set of nodal displacements that satisfies the compatibility requirements for the structure, i.e.,

$[A]^T[u^*] = [\delta^*]$

So [δ*] is still arbitrary, because [u*] is quite arbitrary - we can envision many different sets of member deformations.

With this, our equilibrium equations, pre-multiplied by our arbitrary vector becomes

$[X]^T[u^*] = [f]^T[\delta^*]$ or $[u^*]^T\,[X] = [\delta^*]^T[f]$

(Note: The dimensions of the quantity on the left hand side of this last equation are displacement times force, or work. The dimensions of the product on the right hand side must be the same).

Now we choose [u*] in a special way; we take it to represent a unit, *virtual* displacement associated with a single degree of freedom, all other displacements zero. For example, we take

$[u^*]^T = [\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0......]$

a unit displacement in the vertical direction at node 3 in the direction of $Y_3$.

Carrying out the product $[u^*]^T\,[X]$ in the equation above, we obtain just the external force component associated with the same degree of freedom, $Y_3$ i.e.,

$Y_3 = [\delta^*]^T[f]$

17. We allow the system to be indeterminate as indicated in the figure.

We can cast this last equation into terms of member deformations (and member stiffness) and write:

$Y_3 = [\delta^*]^T [k_{diag}][\delta]$

And that is our special method for determining external forces of a statically determinate (or indeterminate) truss when all displacements are prescribed. It requires, first, solving compatibility for the "actual" member deformations [δ] given the "actual" prescribed displacements. We then solve *another* compatibility problem - one in which we apply a unit, or "dummy" displacement at the node we seek a to determine an applied force component and in the direction of that force component. With the "dummy" member deformations determined from compatibility, we carry out the matrix multiplication of the last equation and there we have it.

We emphasize the difference between the two deformation vectors appearing in this equation; [δ] in plain font, is the vector of actual member deformations in the structure given the actual prescribed nodal displacements. [δ*] starred, on the other hand, is some, originally arbitrary virtual deformation vector which satisfies compatibility - compatibility solution for member deformations corresponding to a unit displacement in the vertical direction at node 3.

We emphasize that our method does not require that we explicitly write out and solve the equilibrium equations for the system. We must, instead, compute compatible member deformations several times over.

## A Generalization

We think of applying Displacement Method #1 at each degree of freedom in turn, and summarize all the relationships obtained for the required applied forces in one matrix equation. We do this by choosing

$$\left[u^*\right]^T = \begin{bmatrix} 1 & 0 & ..... & 0 & 0 \\ 0 & 1 & ..... & 0 & 0 \\ ..... & ..... & ..... & ..... & ..... \\ 0 & 0 & ..... & 1 & 0 \\ 0 & 0 & ..... & 0 & 1 \end{bmatrix} = \left[I\right] \qquad 12 \text{ x } 12$$

where each row represents a unit displacement in the direction of the "row$^{th}$" degree of freedom.

The corresponding member deformations [δ*] now takes the form of a 11 x 12 matrix whose "j$^{th}$"column entries are the deformations engendered by the unit displacement of the "i$^{th}$" row above. (Note there are 11 members, hence 11 deformations and member forces).

We still have

$$[u^*]^T [X] = [\delta^*]^T [f] \qquad \text{where} \qquad [\delta^*]^T = [u^*]^T [A]$$

but now $[u^*]^T$ is a 12 by 12 matrix, in fact the identity matrix. So we can write:

$$[X] = [\delta^*]^T[f]$$

- an expression for all the required applied forces, noting that the matrix $[\delta^*]^T$ is 12 by 11.

Some further manipulation takes us back to the matrix displacement analysis results of the last section. Eliminating $[\delta^*]^T$ via the second equation on the line above, and setting $[u^*]$ to the identity matrix, we have $[X] = [A][f]$

which we recognize as the equilibrium requirement. (But remember, in this world of prescribed displacements, analysts look upon this relationship as foreign; compatibility is their forte). We replace the real member forces in terms of the real member deformations, then, in turn, the real member deformations in terms of the prescribed and actual displacements and obtain

$$[X] = [A][k_{diag}][A]^T[u] \qquad \text{of} \qquad [X] = [K][u]$$

## Design Exercise 5.1

You are a project manager for Bechtel with responsibility for the design and construction of a bridge to replace a decaying truss structure at the Alewife MBTA station in North Cambridge. Figure 1 shows a sketch of the current structure and Figure 2 a plan view of the site. The bridge, currently four lanes, is a major link in Route 2 which carries traffic in and out of Boston from the west. Because the bridge is in such bad shape, no three-axle trucks are allowed access. Despite its appearance, the bridge is part of a parkway system like Memorial Drive, Storrow Drive, et. al., meant to ring the city of Boston with greenery as well as macadam and concrete. In fact, the MDC, the Metropolitan District Commission, has a strong voice in the reconstruction project and they very much would like to stress the parkway dimension of the project. In this they must work with the DPW, the Department of Public Works. The DPW is the agency that must negotiate with the Federal Government for funds to help carry through the project. Other interested parties in the design are the immediate neighborhoods of Cambridge, Belmont, and Arlington; the environmental groups interested in preserving the neighboring wetlands. (Osprey and heron have been seen nearby.) Commuters, commercial interests – the area has experienced rapid development – are also to be considered.

1.1 Make a list of questions of things you might need to know in order to do your job.

1.2 Make a list of questions of things you might need to know to enable you to decide between proposing a four-lane bridge or a six-lane bridge.

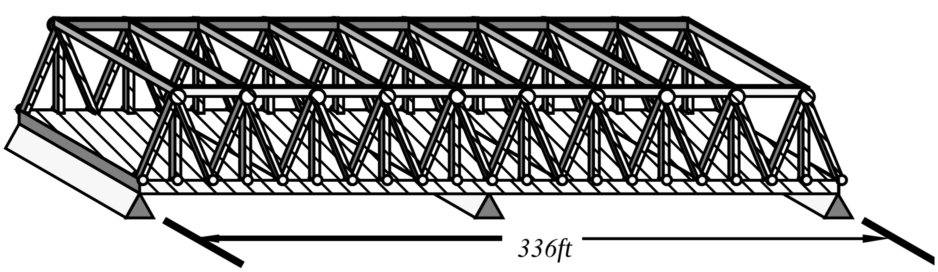
1.3 Estimate the "worst-case" loads a four lane bridge might experience. Include "dead weight loading" as well as "live" loads.

1.4 With this loading:

a) sketch the shear-force and bending-moment diagram for a single span.

b) for a statically determinate truss design of your making, estimate the member loads by sectioning one bay, then another...

c) rough out the sizes of the major structural elements of your design.




.

**CONSERVATION COMMISSION FRUSTRATED AT ALEWIFE PLAN**

(October 4, 1990, Belmont Citizen-Herald) by Dixie Sipher Yonkers, Citizen-Herald correspondent[18]

Opponents of the planned $60 to $70 million Alewife Brook Parkway reconstruction can only hope the federal funding falls through or the state Legislature steps in at the eleventh hours with a new plan. Following a presentation by a Metropolitan District Commission planner on the Alewife Development proposal Tuesday night, the Belmont Conservation Commission expressed frustration over an approval process that appears to railroad a project of questionable benefit and uncertain impact, regardless of communities' concerns and requests. The Alewife project would widen Route 2 and redesign the truss bridge, access roads and access ramps on Route 2 near the Belmont-Arlington-Cambridge border. It also would extend Belmont's Brook Parkway significantly. Alewife Basin planner John Krajovick told the commission that MDC has grave concerns about the proposed transportation project and that, funding issues aside, it might be impossible to prevent the Massachusetts Department of Public Works' "preferred alternative" from being implemented. According to Krajovick, the MDC's concerns center around the loss of open space that will accompany the project, specifically the land along the eastern bank of Yates pond, the strip abutting the existing parkway between Concord Avenue and Route 2, the wetlands along the railroad right-of-way near the existing interim access road, and that surrounding the Jerry's pool site. "Our goal is to reclaim parkways to the original concept of them," said Krajovick. "It was Charles Elliot's vision to create a metropolitan park system – a kind of museum of unique open spaces...and use the parkways to connect them as linear parks." "The world has changed. They are no longer for pleasure vehicles only, but parkways, we feel, are a really important way to help to control growth and maintain neighborhood standards," he added. "We would like to see the character of this more similar to Memorial Drive and Storrow Drive as opposed to an expressway like Route 2." Krajovick outlined the MDC's further concerns with the project, citing its likely visual, physical, noise, and environmental impacts on surrounding neighborhoods. Projected to cost $60-$70 million, he said, the "preferred alternative" will also hurt a sensitive wetland area, the Alewife Reservation, in return for minimal traffic improvements. In spite of these concerns, Krajovick reported that the project is nearing a stage at which it becomes very difficult to prevent implementation. The Final Environmental impact Statement is expected to be submitted to the Federal Highway Department within a month. The same document will be used as the final report the state's Executive Office of Environmental Affairs. EOEA Secretary John DeVillars cannot stop the project once he receives that report. He can call for mitigating measures only. Krajovick noted a bill currently before the state Legislature's Transportation Committee could prohibit the project from going forward as presently designed. He took no position on that bill. Conservation Commission members, however, voiced doubts

18. Reprinted with permission of Harte-Hanks Community Newspapers, Waltham, MA

on the likelihood a passage in the face of the fiscal crisis and state elections that loom before legislators. In addition, Krajovick said that state budget cuts are expected to result in layoffs for nearly 600 of the MDC's staff of 1,000 workers, effectively decimating the agency. "Our hopes for a compromise solution may not happen," he said. Discouraged by Krajovick's dismal prognosis, Conservation Commission members expressed concern that there was nothing they could do to change the course of the project. The commission has been providing input on the project for 12 years with no results. In response to Krajovick's presentation, Commission member William Pisano called the need for updated impact studies, saying, "We agree with you. What we want to see is a lot more data and a more accurate realization of what we're playing ball with today." Commending the way in which concerned residents of Arlington, Cambridge and Belmont have gotten involved in the project, however, Krajovick said their thinking as a neighborhood rather than individual towns is a positive thing that has come from the project. Building on that team spirit, he said, the communities can raise their voice through formation of a Friends group and work toward the development of a master plan or restoration plan for the whole Alewife reservation area.

**BRIDGE MEETING HIGHLIGHTS ISSUES**

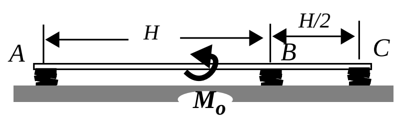
Belmont Citizen-Herald September 26, 1991 by Alin Kocharians, Citizen-Herald staff

Some 50 residents turnout out Tuesday night at Winn Brook School to hear a presentation by the state Metropolitan District Commission on the Alewife Brook Parkway Truss Bridge. MDC representatives previewed their Truss Bridge renovation and Parkway restoration plans. The Parkway segment affected is in Cambridge, between the Concord Avenue rotary and Rindge Avenue. Plans for the two-year project, MDC officials hope, will be completed by early 1992, with construction following in the spring of that year. Julia O'Brien, MDC's director of planning, said that the $12 million necessary for the project will be provided by the Legislature and federal grants. Once the bridge renovation is completed, the truck ban on it will be lifted, hopefully reducing truck traffic in Belmont. The renovation plans are 75 percent complete, according to John Krajovic, the MDC planner in charge of the project. The MDC is also visiting with Arlington and Cambridge residents, asking for input on the project's non-technical aspects. Residents and MDC representatives exchanged compliments in the first hour, but as the meeting wore on, the topics of cosmetic versus practical and local versus regional issues proved divisive. One Belmont resident summed up what appeared to be a common misgiving in town. "I don't want to cast stones, because it is a nice plan," said John Beaty of Pleasant Street, "but it doesn't solve the overall problem. I wish that I were seeing not just MDC here. There were two competing plans. It is the (State Department of Public Works') charter to solve the overall region's problem. I see those two as being in conflict." Beaty said that the DPW plan was presented two years ago to residents, when officials had said that the plan was 60 percent complete. Stanley Zdonik of Arlington agreed. "I am impressed with the MDC presentation, but what bothers me is, are you going to improve on the traffic flow?" he said. "You have got one bottleneck at one end, and another at the other." He said that the Concord Avenue and Route 2 rotaries at either end of the bridge should have traffic signals added, or be removed altogether. Krajovic replied that according to what the MDC's traffic engineer had told him, "historically, signalizing small rotaries actually backs up traffic even more." Belmont Traffic Advisory Committee member Marilyn Adams took issue with the decision not to add signals to the rotaries, and asked to see the study that produced this recommendation. Adams was also concerned with a "spill off" of traffic from the construction. "I can't guarantee people won't seek out other routes," including Belmont, O'Brien said. However, she added, she did not expect the impact to be very great, as the Parkway would still be open during construction. "We will make really a strong effort for a traffic mitigation" plan to be negotiated with the town, she said. Selectman Anne Paulsen also asked about the impact of traffic on the town. MDC representatives said that various traffic surveys were being conducted to find a way to relieve the traffic load on Belmont. Krajovic said that traffic problems in Belmont were regional questions, to be handled by local town officials, a point with which Paulsen disagreed. Paulsen said that she would prefer a more comprehensive plan for the region. Aside from the reconstruction of the Truss Bridge, she said, "I think the point of the people of Belmont is that...we want improvement in the roadway, so that we are relieved of some of the traffic." According to the plans, the new bridge will have four 11-foot lanes, one foot wider than the current width for each lane. There will also be a broader sidewalk, and many new trees planted both along the road and at the rotaries. There will be pedestrian passes over the road, and a median strip with greenery. The bridge will be made flat, so that motorists will have better visibility, engineering consultant Ray Oro said. It will be constructed in portions, so that two lanes will always be able to carry traffic, he said. According to Blair Hines of the landscaping firm of Halvorson Company, Inc., by the end of the project, "Alewife Brook Parkways will end up looking like Memorial Drive." All the talk about landscaping, Paulsen suggested with irony, "certainly calms the crowd."

## 5.4 Problems

**5.1** If the springs are all of equal stiffness, k, the bar ABC rigid, and a couple Mo is applied to the system, show that the forces in the springs are

$$F_A = -(5/7)Mo/H \qquad F_B=(1/7)M_o/H \qquad F_C=(4/7)M_o/H$$

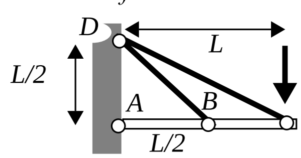



**5.2** The problem show within the box was worked incorrectly by an MIT student on an exam. The student's work is shown immediately below the problem statement, again with the box.

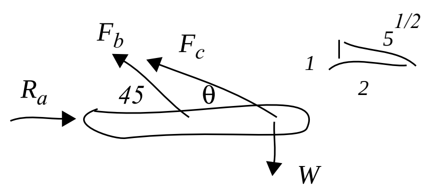
i) Find and describe the error.

ii) Re-formulate the problem—that is, construct a set of equations from which you might obtain valid estimates for the forces in the two supporting members, *BD* and *CD*.

*A* **rigid** *beam is supported at the three pins, A,B, and C by the wall and the two elastic members of common material and identical cross-section. The rigid beam is weightless but carries an end load W. Find the forces in the members BD and CD in terms of W.*




θ=*arctan(1/2)*




1 $R_a - F_b\sqrt{2}/2 - F_c\,2/\sqrt{5} = 0$

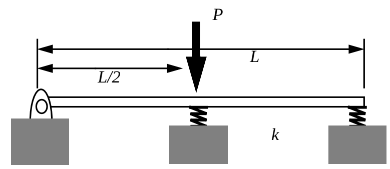
2 $-W + F_b\sqrt{2}/2 + F_c/\sqrt{5} = 0$

3 $F_b\sqrt{2}/2 \; L/2 + F_c\, L/\sqrt{5} - WL = 0$

*rewrite 3* $-W + F_b\sqrt{2}/4 + F_c/\sqrt{5} = 0$

*subtract 2* . $F_b = 0$ *and* $F_c = \sqrt{5}\, W$ *ans.!*

**5.3** A *rigid* beam is pinned supported at its left end and at midspan and the right end by two springs, each of stiffness $k$ (force/displacement). The beam supports a weight P at mid span.




i) Construct a compatibility condition, relating the displacements of the springs to the rotation of the rigid beam.

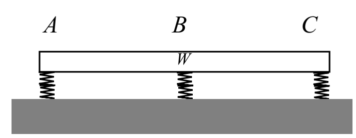
ii) Draw an isolation and write out the consequences of force and moment equilibrium

iii) Using the force/deformation relations for the linear springs, express equilibrium in terms of the angle of rotation of the beam.

iv) Solve for the rotation, then for the forces of reaction at the three support points.

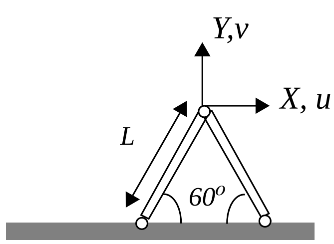
v) Sketch the shear force and bending moment diagram.

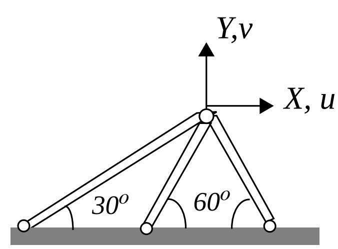
**5.4** For the rigid stone block supported by three springs of Exercise 5.1, determine the displacements of and forces in the springs (in terms of $W$) if the spring at $C$ is very, very stiff relative to the springs (of equal stiffness) at $A$ and $B$.




**5.5** The stiffness matrix for the truss structure shown below left is

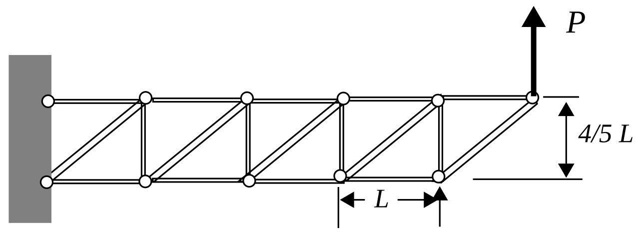
$$2\left[\frac{AE}{L}\right]\begin{bmatrix} cos^2 60 & 0 \\ 0 & sin^2 60 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}$$



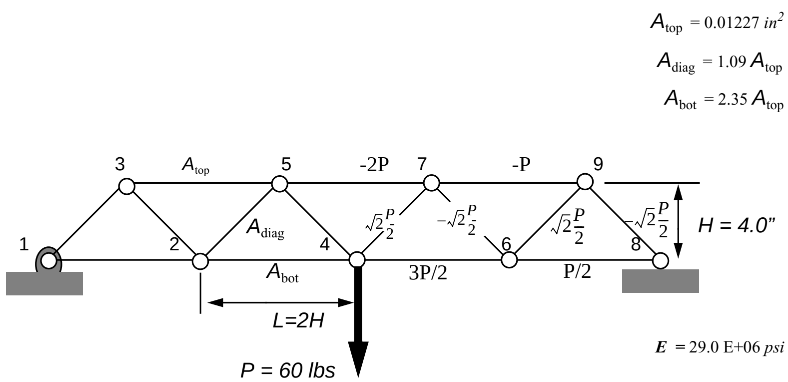




*What if* a third member, of the same material and cross-sectional area, is added to the structure to *stiffen it up*; how does the stiffness matrix change?

**5.6** Without writing down any equations, *estimate* the maximum member tensile load within the truss structure shown below. Which member carries this load?



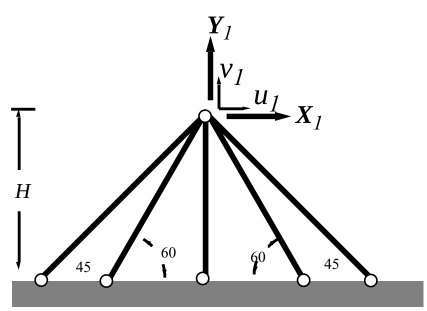

**5.7** The truss show below is loaded at midspan with a weight P= 60 lbs. The member lengths and cross sectional areas are given in the figure. The members are all made of steel.




a) Verify that the forces in the members are as indicated.
b) Using Trussworks, determine the vertical deflections at nodes 2 and 4.

**5.8** All members of the truss structure shown at the left are of the same material (Elastic modulus E), and have the same cross sectional area. Fill in the elements of the stiffness matrix.

$$\left(\frac{AE}{H}\right)\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}$$

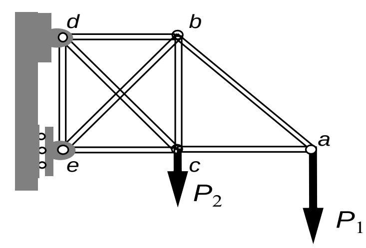


**5.9** For the three problems 1a, 1b,and1c, state whether the problem posed is statically *determinate* or statically *indeterminate.* In this, assume all information regarding the geometry of the structure is given as well as values for the applied loads.

1a) Determine the force in member *ab*.

1b) Determine the force in member *bd*

1c) Determine the reactions at the wall.



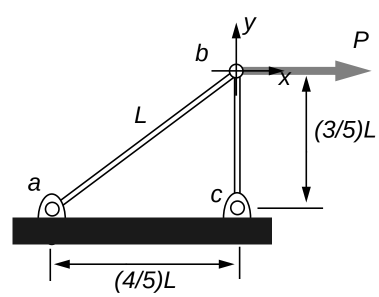

**5.10** The simple truss structure shown is subjected to a horizontal force P, directed to the right. The members are made of the same material, of Young's modulus E, and have the same cross-sectional area, A (for the first three questions).

i) Find the force acting in each of the two members *ab, bc*, in terms of P.

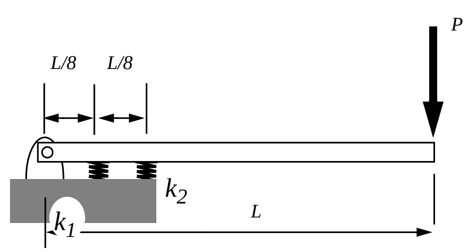
ii) Find the extension, (contraction), of each of the two members.

iii) Assuming small displacements and rotations, sketch the direction of the displacement vector of node *b.*

iv) Sketch the direction of the displacement vector if the cross-sectional area of *ab* is much greater than that of *bc*.




v) Sketch the direction of the displacement vector if the cross-sectional area of *ab* is much less than that of *bc*.




**5.11** The rigid beam is pinned at the left end and supported also by two linear springs as shown.

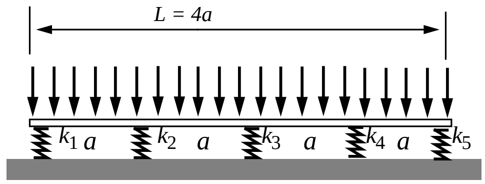
What do the equilibrium requirements tell you about the forces in the spring and their relation to P and how they depend upon dimensions shown?

Assuming small deflections (let the beam rotate *cw* a small angle $\theta$), what does compatibility of deformation tell you about the relationships among the contractions of the spring, the angle $\theta$?

What do the constitutive equations tell you about the relations between the forces in the springs and their respective deflections?

Express the spring forces as a function of P if $k_2 = (1/4)k_1$

**5.12** A rigid board carries a uniformly distributed weight, W/L. The board rests upon five, equally spaced linear springs, but each of a different stiffness.




Show that the equations of equilibrium for the isolated, rigid board can be put in the form

$$[A] \cdot [F] = \begin{bmatrix} W \\ 0 \end{bmatrix}$$

where [A} is a 2 by 5 matrix and [F] is a 5 by 1 column matrix of the compressive forces in the five springs. Write out the elements of [A}.

If the springs are linear, but each of a different stiffness, show that the matrix form of the force/deformation relations take the form

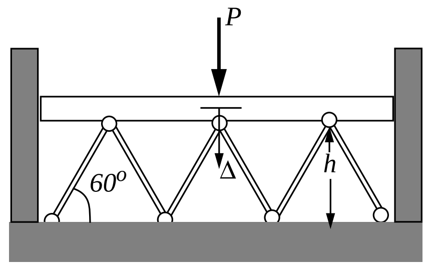
$$[F] = [k_{\text{diag}}] \cdot [\delta]$$

where the [δ] is a the column matrix of the spring deformations, taken as positive in compression, and the k matrix is diagonal.

Show that, if the beam is rigid and deformations are small then, in order for the spring deformations to be compatible, one with another, five equations must be satisfied (for small deformations). Letting $u$ be the vertical displacement of the midpoint of the rigid beam - positive downward - and θ its counter-clockwise rotation, write out the elements of $[A]^T$ - the matrix relating the deformations of the spring to $u$ and θ. Then show that the equations of equilibrium in terms of displacement take the form:

$$[A] \cdot [k_{\text{diag}}] \cdot [A]^T \cdot \begin{bmatrix} u \\ \theta \end{bmatrix} = \begin{bmatrix} W \\ 0 \end{bmatrix}$$

where $[A]^T$ is the transpose of [A].




**5.13** A rigid beam is constrained to move vertically without rotation. It is supported by a simple truss structure as shown in the figure. The truss members are made of aluminum (*E = 10.0 E+06 psi; = 70 GPa*). Their cross sectional area is 0.*1 in*$^2$ *= 0.0645E-03 m*$^2$ The system is bears a concentrated load *P* at mid span. The height of the platform above ground is *h =36 in = 0.91 m*.

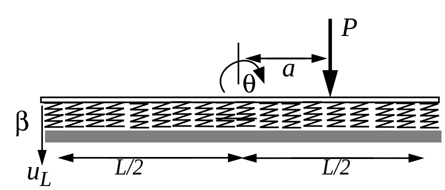
Let the vertical displacement be Δ. Determine the value of the stiffness of the system, the value for K in the relationship $P = K\Delta$

What is the vertical displacement if P = 5,000 lb = 22,250 N

What is the compressive stress in the members at this load?

**5.14** A rigid beam rests on an elastic foundation. The distributed stiffness of the foundation is defined by the parameter β; the units of β are force per vertical displacement per length of beam. (If the beam were to displace downward a distance $u_L$ without rotating, the total vertical force resisting this displacement would be just $\beta * u_L * L$). A heavy weight *P* rests atop the beam at a distance *a* to the right of center. The beam has negligible weight relative to *P*.

Letting the vertical displacement at the *left end* of the beam be $u_L$, and the rotation about this same point be θ, (clockwise positive), show that the requirements of force and moment equilibrium, applied to an isolation of the beam, give the following two equations for the displacement and rotation:




$$(\beta L)\cdot u_L + \left(\frac{\beta L^2}{2}\right)\cdot\theta = P$$

$$\left(\frac{\beta L^2}{2}\right)\cdot u_L + \left(\frac{\beta L^3}{3}\right)\cdot\theta = P\cdot(a + L/2)$$

Let $\Lambda = P/(\beta L^2)$, $\alpha = a/L$, and $z = u_L/L$ so to put these in non dimensional form. Then solve for the non-dimensional displacements *z* and θ.Explore the solution for special cases, e.g., *a = 0, -L/2, +L/2*. What form do the equilibrium equations take if you measure the vertical displacement at the center of the beam? (Let this displacement be $u_o$).