1.050: Deformation and Strain Tensor (HW#7)

Due: October 31, 2007

Team Building and Team Work: We strongly encourage you to form Homework teams of three students. Each team only submits one solution for correction. We expect true team work, i.e. one where everybody contributes equally to the result. This is testified by the team members signing at the end of the team copy a written declaration that "the undersigned have equally contributed to the homework". Ideally, each student will work first individually through the homework set. The team then meets and discusses questions, difficulties and solutions, and eventually, meets with the TA or the instructor. Important: Specify all resources you use for your solution.

The following set of exercises is designed to familiarize you with deformation and strain measurements.

1. Derivation: Show the derivation of:

$$J = \det \underline{\underline{F}}$$

for the special case of a volume change in only two directions. From the lecture notes, we derived that the deformed volume $d\Omega_d$ is related to the original volume $d\Omega_0$ by:

$$d\Omega_d = \left(\underline{\underline{F}} \cdot d\vec{X}_1\right) \cdot \left(\underline{\underline{F}} \cdot d\vec{X}_2 \times \underline{\underline{F}} \cdot d\vec{X}_3\right) = \det \underline{\underline{F}}\left[d\vec{X}_1 \cdot \left(d\vec{X}_2 \times d\vec{X}_3\right)\right] = J d\Omega_0$$

where $d\Omega_0 = d\vec{X}_1 \cdot \left(d\vec{X}_2 \times d\vec{X}_3\right)$. For the 2-D case, the deformation gradient $\underline{\underline{F}}$ and the material vectors are:

$$\underline{\underline{F}} = \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \vec{X}_1 = \begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} \quad ; \quad \vec{X}_2 = \begin{bmatrix} a_2 \\ b_2 \\ 0 \end{bmatrix} \quad ; \quad \vec{X}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2. A 'Simple' Shear Exercise: In this exercise, we will investigate the deformation of a cube by simple shear. The cube has sides of unit-length. The magnitude of the simple shear is given on the sketch below.
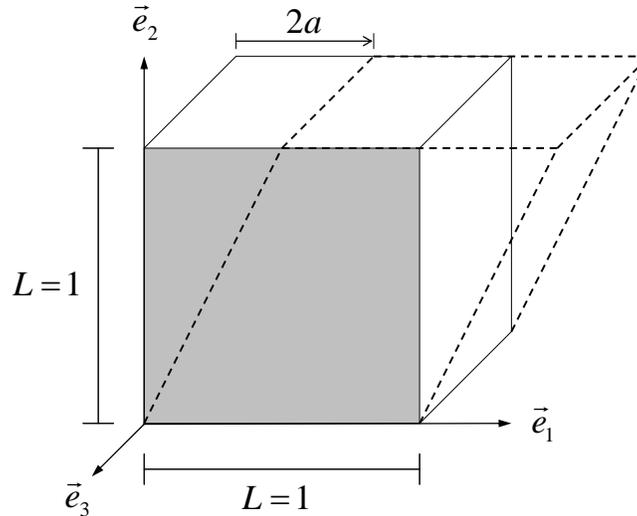
The displacement field for the simple shear is defined by:

$$\vec{\xi}(x, y, z) = (2ay)\vec{e}_1$$

Where y is the distance in the $\vec{e}_2$-axis.

a. FINITE DEFORMATION THEORY: From finite deformation theory (refer to manuscript and lecture notes), the following results are obtained:

$$\text{Change in volume: } \frac{d\Omega_t - d\Omega_0}{d\Omega_0} = 0$$

$$\text{Change in surface oriented by } \vec{N} = \vec{e}_1 : \ da = \sqrt{1 + 4a^2}\, dA$$

$$\text{Maximum dilation: } \lambda(\vec{u}_I) = \sqrt{2a^2 + 2\sqrt{a^4 + a^2 + 1}} - 1$$

$$\text{Distortion in plane } (\vec{e}_1 \times \vec{e}_2): \ \theta(\vec{e}_1, \vec{e}_2) = \sin^{-1}\left(\frac{2a}{\sqrt{1 + 4a^2}}\right)$$

For this part of the exercise, use your geometry skills <u>only</u> to calculate:
  i.   The change in volume $(d\Omega_t - d\Omega_0)/d\Omega_0$
  ii. The change in surface da for the surface oriented by $\vec{N} = \vec{e}_1$
  iii. The distortion $\theta(\vec{e}_1, \vec{e}_2)$
  iv. Do your results match with the results from finite deformation theory?
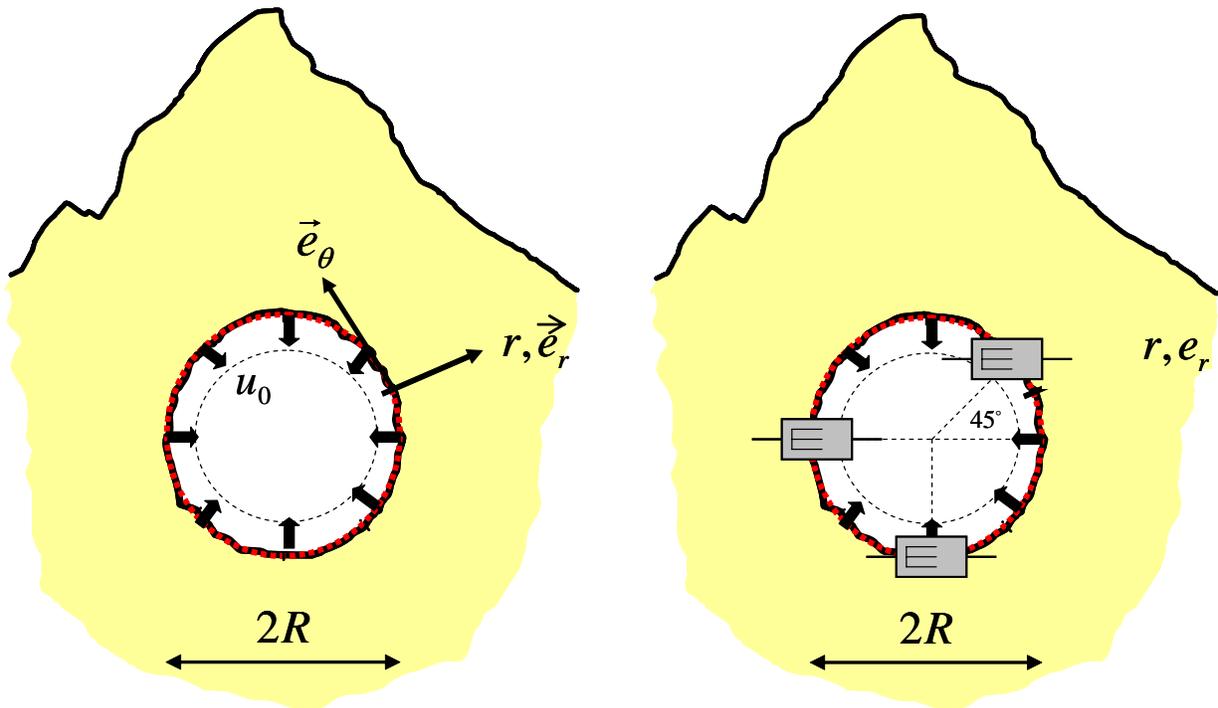
b. INFINITESIMAL (LINEAR) DEFORMATION THEORY:

    i.   Under what condition can we restrict our study of simple shear to linear deformation theory? Recall that $\left\| Grad\,\vec{\xi} \right\| = tr\left(Grad\,\vec{\xi} \cdot \left(Grad\,\vec{\xi} \right)^{T} \right)$, where '$tr$' is the trace of the matrix.

    ii. Assuming that linear deformation theory is valid, calculate the strain tensor $\underset{=}{\varepsilon}$.

    iii. Draw the Mohr circles for the result of part ii). In what direction do you get maximum dilation? What is the value of the maximum dilation?

    iv. Compute the same quantities required in parts a) i-iii using now infinitesimal deformation theory. Discuss your results.

c. Numerical Application: Using the previous results from finite and infinitesimal deformation theories, construct plots for:

    i.   Maximum dilation, $\lambda(\vec{u}_{I})$

    ii. Distortion in plane $(\vec{e}_{1} \times \vec{e}_{2})$, $\theta(\vec{e}_{1}, \vec{e}_{2})$

as functions of length parameter a. Use a scale between $0 < a < 0.5$. Comment on your results and provide an estimated range over which linear deformation theory could be applied for the single shear deformation.

3. Tunneling: During the excavation of a tunnel, one often observes that the walls of the tunnel
tend to move inward, such as displayed on the following sketch (on the left side).

This phenomenon is called *convergence*. Given the symmetries of the problem, we decide to
work in cylindrical coordinates. We assume that the displacement field is given by:

$$\vec{\xi}(r,\theta,z) = -\frac{u_0 R}{r}\vec{e}_r$$

a. Under what condition can we restrict our study to the linear deformation theory? (Use the
   same approach as in Problem 2, Part b,i).
b. Assuming that the linear deformation theory is valid, calculate the strain tensor $\underline{\underline{\varepsilon}}$.
c. What is the volume strain?
d. What are the eigenvalues and eigenvectors? Using these results, draw the Mohr circles.
e. In what direction do you get maximim dilation? In what direction do you get maximim
   distortion?
f. To measure the strains, strain gauges are placed in the wall of the tunnel. Their
   arrangement is displayed in the top figure (on the right side). In view of the given
   displacement field, what <u>dilation</u> do you expect each of the strain gauges to measure?

4. Three-Truss System: Consider the truss system displayed in the following figure. The structure is composed of three bars of the same length L, supported at the top and connected on the bottom side to a <u>rigid</u> beam device, which is subjected to a load P at location $x = (3/4)l$. In a first approach, this truss system can be considered as a discrete system. The objective of this problem is to determine the axial force in each truss member and evaluate the displacement ξ_0 at the point of load application.
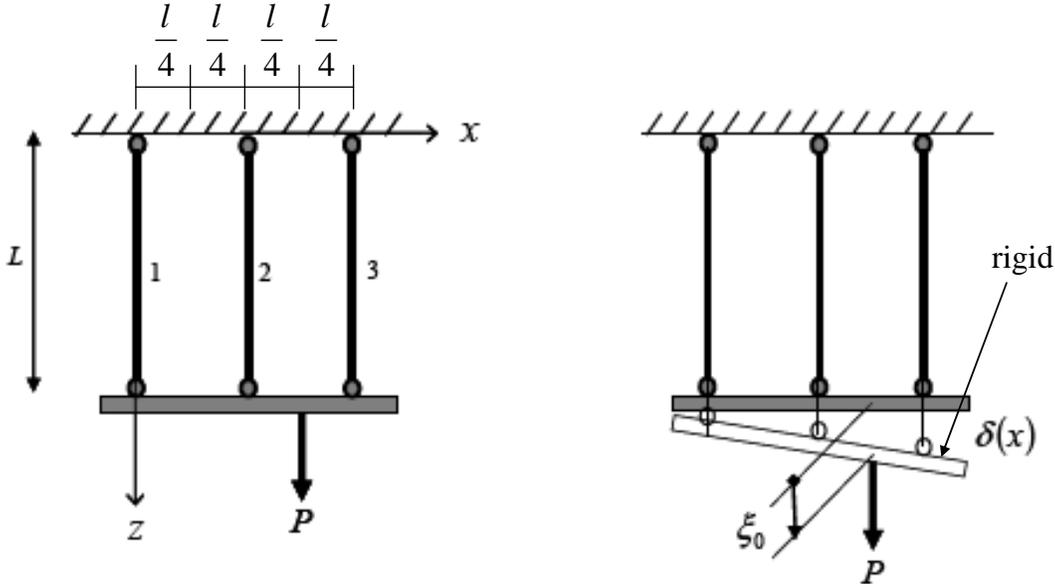
The solution of this problem requires the following steps:

a. Apply force and moment equilibrium (around the point of load application), and derive the two equations relating the axial forces N_1, N_2, N_3 and load P (Note that the axial force N_i corresponds to bar $i = 1$, 2, 3).

b. From the result in part a), what is the degree of indeterminacy?

c. In order to solve the given problem, we need to ensure the kinematic compatibility between the bar elongations δ_i (which correspond to the elongations of bars $i = 1$, 2, 3) and the constraints on both the top (where the bars are fixed) and the bottom where each bar is constraint by the displacement ξ_i of the rigid bar at the nodes. The geometrical compatibility conditions are:

$$\delta_1 = \xi_1$$

$$\delta_2 = \xi_2 = \xi_1 + \frac{2}{3}(\xi_0 - \xi_1)$$

$$\delta_3 = \xi_3 = \xi_1 + \frac{4}{3}(\xi_0 - \xi_1)$$

Explain how the compatibility conditions were derived.

d. There exists a relation between the elongation and the force in each bar given by:

$$N_i = K_S \delta_i$$

where δ_i corresponds for the elongation of bar i. This offers additional equations to eliminate the indeterminacy of the problem. Substitution of these relations and the geometrical compatibility conditions in the equilibrium relations obtained in part a), determine the displacements ξ_0 and ξ_1.

e. Finally, using these results, determine the axial forces in the three truss elements.

5. Variation of Quiz 1

To provide additional practice, please redo a slightly modified version of Quiz I.

A cylindrical pothole in a pavement is filled with a new material subjected to vehicle traffic. The aim of this exercise is to analyze stresses and strength compatibility of this repair solution.

Engineering Model: To this end, we model the pothole as a cylinder of diameter and height L (see figure below). The surrounding pavement (old material) is so old that it can be considered as rigid compared to the new material filling the cubic pothole. The load to which the pot hole is subjected is the tire pressure p. We neglect body forces (*i.e.* deadweight), and all surfaces between the new material in the pothole and the rigid material surrounding it, are frictionless. The entire problem will be solved in a cylindrical coordinate system.
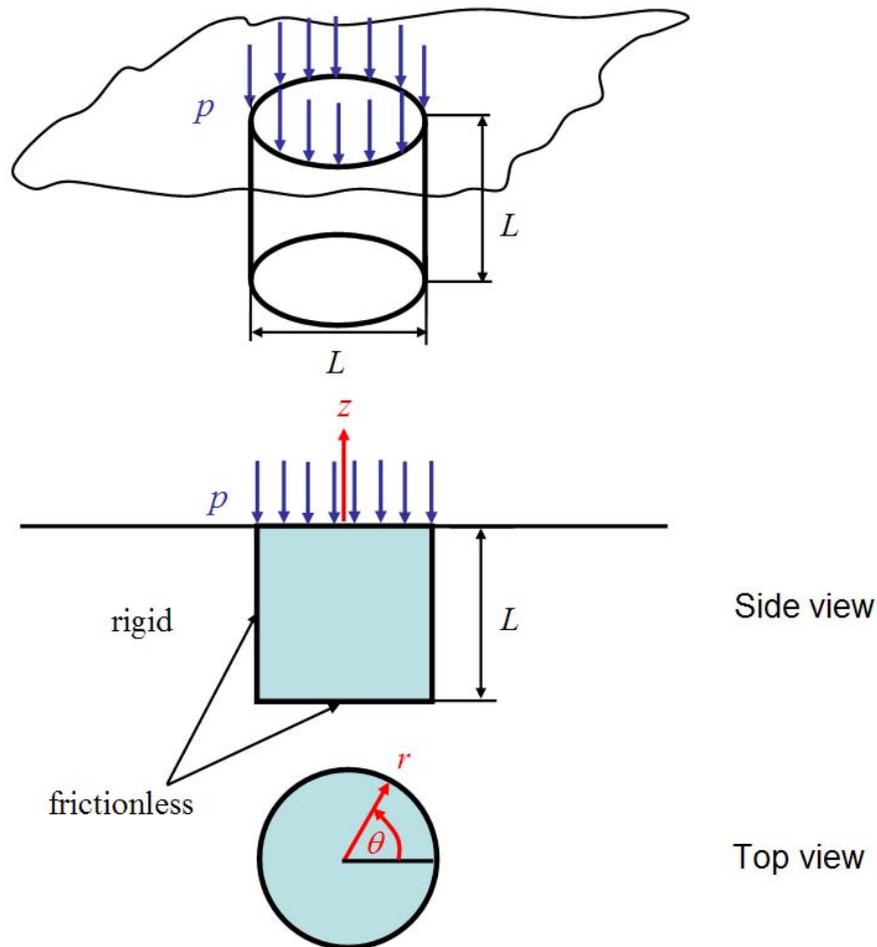
5.1 Statically admissible stress field

The goal of the first part of this exercise is to calculate the stress field inside the pothole.

a. Formulate precisely all conditions the stress field inside the pothole needs to satisfy to be
    statically admissible (S.A.). In particular:

    i.   Formulate the stress vector boundary condition (that is, $\vec{T}^d$) on the surface of the
         pothole.

    ii. Formulate the stress vector boundary condition on the frictionless surface between the
         pothole and the rigid material, as well as at the bottom of the pothole.

    iii. Assuming that the stresses are constant in the pothole (neglect gravity), show that the
          stress field is compatible with a diagonal form of the stress tensor ($\sigma_{rr}=\sigma_{\theta\theta}$ due to
          equilibrium):

$$\underline{\underline{\sigma}} = a(\vec{e}_r \otimes \vec{e}_r + \vec{e}_\theta \otimes \vec{e}_\theta) + b\,\vec{e}_z \otimes \vec{e}_z.$$

where a and b are two unknown stress parameters.

b. The stress field has two parameters, a and b. Which of these two parameters can be
    determined from the analysis done in part a)? Determine this parameter. (The other
    parameter remains unknown; we will determine it in part 5.2).

5.2 Mohr circle plot and strength criterion

a. Sketch the Mohr circle, indicating the parameters a and b in the Mohr plane (assume that
    $a > b$).

b. The pothole is filled with concrete. The strength of concrete can be approximated by
    using a Tresca strength model (this is a rough approximation, but considered sufficient
    for the present analysis). In terms of principal stresses it reads (see lecture notes):

$$f(\underline{\underline{\sigma}}) = \sigma_I - \sigma_{III} - 2c \leq 0$$

    i.   Define the condition at the strength limit.

    ii. Determine the three principal stresses $\sigma_{I,II,III}$ based on the stress field given in
         part 5.1).

    iii. Using the Tresca strength model, find an expression for the missing stress field
          parameter (a or b), assuming that the strength limit is reached under the load p (this
          expression will be a function of p as well as the Tresca parameter c.

        Hint: precisely define the condition at the strength limit.

5.3 Application of strength criterion, numerical values

a. The Tresca strength model contains one material parameter, the cohesion c. The goal of this question is to determine this parameter from a laboratory experiment. The <u>laboratory</u> experiment that is carried out on concrete specimens made of the same material as used to fill the pothole is the uniaxial tensile test. From the experiments, the uniaxial tensile strength of $\sigma_t = 3$ MPa was determined.

By carefully determining the principal stresses in the laboratory test, determine cohesion c of the concrete. In particular:

i.   To facilitate your calculation, first sketch the Mohr circle for the test (Hint: first determine σ_I and σ_{III} for the uniaxial tension test).

ii. Sketch the Tresca strength criterion in the Mohr plane in the situation when the strength limit in uniaxial tension is reached.

iii. By using the mathematical expression of the Tresca strength criterion, determine the cohesion c.

Note: No graphical analysis is needed to obtain the answer for part iii); points i) and ii) are only to visualize the concepts. This exercise can be solved independently.

b. The typical axle load (4 tires) of a US Highway truck is 400 kN (equivalent to 40 metric tons), which is transmitted over a tire contact surface of roughly $A{=}200$ cm^2. Determine the values of the stress tensor components.

For these values, is the vertical interface between the old and the new material subjected to tension or compression?