1.010 Uncertainty in Engineering
Fall 2008

Example Application 12

(Functions of random variables and reliability analysis)

DISTRIBUTION OF WAVES AND WAVE LOADS IN A RANDOM SEA

The Ocean Surface and the Distribution of Wave Height

The elevation of the sea surface during a storm is adequately described by what is called a normal (or Gaussian) random function. This means that, at a generic location and time, the sea surface elevation has with good approximation a normal distribution around the mean sea elevation. Considerations from the theory of normal random functions lead to a distribution of wave height H of the Rayleigh type, with CDF

$$F_H(h) = 1 - e^{-2(h/H_s)^2} \tag{1}$$

where the parameter H_S, called the significant wave height, is four times the standard deviation of the sea surface elevation at a generic point. H in Eq. 1 has mean value 0.627 H_S and standard deviation 0.328 H_S.

Curve (a) in Figure 1 is a plot of the Rayleigh probability density $f_X(x) = xe^{-x^2/2}$ of the normalized variable $X = 2H/H_s$. Our main interest is in the upper tail of the distribution of H, because the larger waves are those that pose threat to ships, offshore structures and marine operations in general.

Maximum Wave Height in a Storm

$F_H(h)$ in Eq. 1 is the probability that a wave chosen at random in a sea state with parameter H_S has height less than h. Consider now a storm with 1000 waves (this number of waves is representative of a storm having an effective duration of 3 hours and an average wave period of 10 seconds) and suppose that such waves have independent and identically distributed heights. Then the distribution of the maximum wave height during the storm, say H_{1000}, is

$$
\begin{aligned}
F_{H_{1000}}(h) &= \{F_H(h)\}^{1000} \\
&= \left\{1 - e^{-2(h/H_s)^2}\right\}^{1000}
\end{aligned}
\qquad (2)
$$

The probability density function of $X_{1000} = 2H_{1000}/H_s$ is shown in Figure 1, as curve (b). For 1000 waves, the most probable maximum height is found to be $1.868H_S$. The distribution of maximum wave height is not very sensitive to the storm duration. For example, for a storm with 2000 waves, the most probable maximum wave height is $1.958H_S$; see curve (c) in Figure 1, which shows the probability density function of $X_{2000} = 2H_{2000}/H_s$.

Maximum Wave Load During a Single Storm

For engineering design or safety verification, one is concerned not just with the distribution of wave height, but also with the distribution of the resulting load on the structure of interest. A simple model for the load q applied by a wave of height h to a fixed platform is

$$
q \propto h^C \qquad (3)
$$

where C is a constant whose value is close to 2. Therefore, if one denotes by h_0 the design wave height and by q_0 the associated design force, the relation between q and h is

$$
\begin{aligned}
q &= q_0\,(h/h_0)^C \\
h &= h_0\,(q/q_0)^{1/C}
\end{aligned}
\qquad (4)
$$

Since the relationship between q and h is monotonically increasing, the distributions of wave height H and load Q are related as

$$F_Q(q) = F_H[h(q)] \\ = F_H\left[h_o(q/q_o)^{1/C}\right] \tag{5}$$

Using for F_H the distribution in Eq. 1 or 2, one obtains the distribution of the maximum load Q due to one or 1000 waves, in a sea with significant wave height H_S. These load distributions are

$$F_Q(q) = 1 - e^{-2(h_o/H_s)^2(q/q_o)^{2/C}} \tag{6a}$$

$$F_{Q_{1000}}(q) = \{F_Q(q)\}^{1000} \\ = \left\{1 - e^{-2(h_o/H_s)^2(q/q_o)^{2/C}}\right\}^{1000} \tag{6b}$$

A plot of the exceedance probability $P[Q > q] = e^{-2(h_o/H_s)^2(q/q_o)^{2/C}}$ against q/q_o for a single wave is shown as curve (a) in Figure 2, where it has been assumed that C = 2 and $h_0 = 1.868H_S$ (as was said before, this value of h_o corresponds to the most probable wave height in a 1000-wave storm with significant wave height H_S). The exceedance probability $P[Q_{1000} > q] = 1 - \left\{1 - e^{-2(h_o/H_s)^2(q/q_o)^{2/C}}\right\}^{1000}$ during a 1000-wave storm is plotted as curve (b). Curve (b) shows that there is a probability about 60% that the design load is exceeded during a 1000-wave storm with the value of H_S used for design. Curve (b) also shows that during the "design storm" there is a probability about 5% that the maximum load on the structure will exceed the design load by 40%.

Maximum Annual Load

Finally, consider the distribution of the maximum annual load Q_{year} on a platform at a certain site, say in the Gulf of Mexico. The main threat to the platform comes from hurricanes. In order to obtain the distribution of the annual maximum H_S at the site, H_{Smax}, one typically uses a simulation approach in which synthetic hurricane occurrences and their paths are numerically generated together with their intensity, size and speed parameters. A sea state model is then used to predict the temporal evolution of H_S at the site for each simulated hurricane. From simulations covering a long period of time, the distribution of H_{Smax} is then obtained. For example, Figure 3 shows the distribution of H_{Smax} derived from 1600 years of simulation at a site south of Galveston, in 600 feet of water. A distribution of the following (truncated exponential) type fits the empirical data well:

$$
\begin{aligned}
1 - F_{H_{smax}}(h) &= 0.47, && \text{for } 0 < h \le 4.14\text{m} \\
&= 3.11 e^{-h/2.19}, && \text{for } h > 4.14\text{m}
\end{aligned}
\qquad (7)
$$

Notice that 1 – 0.47 = 0.53 is accepted to be the probability that $H_{Smax} = 0$, i.e. the probability that, in a generic year, the site does not experience hurricanes.

Problem 12.1

(a) Calculate the mean value of H_{smax} in the above equation, $E[H_{Smax}]$.

(b) Combine the distribution of Q_{1000} for given H_S in Eq. 6b for $C = 2$ and $h_0 = 3E[H_{Smax}]$ with the distribution of H_{Smax} in Eq. 7 to obtain the distribution of the maximum normalized load Q_{1000}/q_0 in one year (assuming that the maximum load occurs during the hurricane with significant wave height H_{Smax}). What is the probability of exceeding the design load q_0 in one year?

(c) Suppose that the safety goal is to design the platform for a load that is exceeded with probability 0.01 in one year. Modify the above design wave height h_0 to achieve this safety goal.

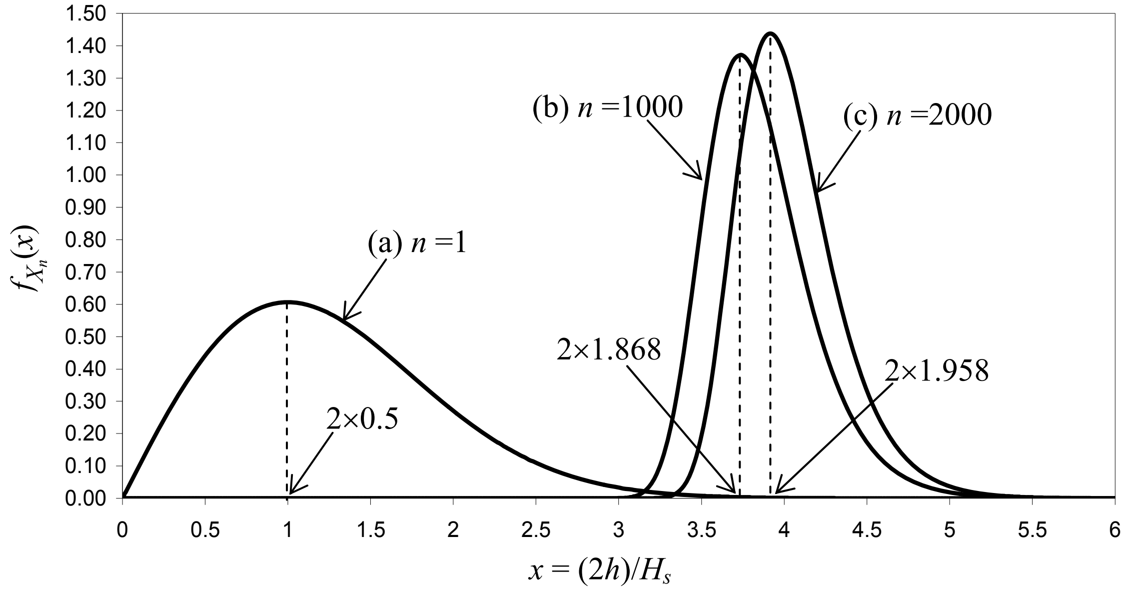

Figure 1: Probability density function, $f_{X_n}(x)$, of the maximum normalized height, $X_n = (2H_n)/H_s$, of n waves.

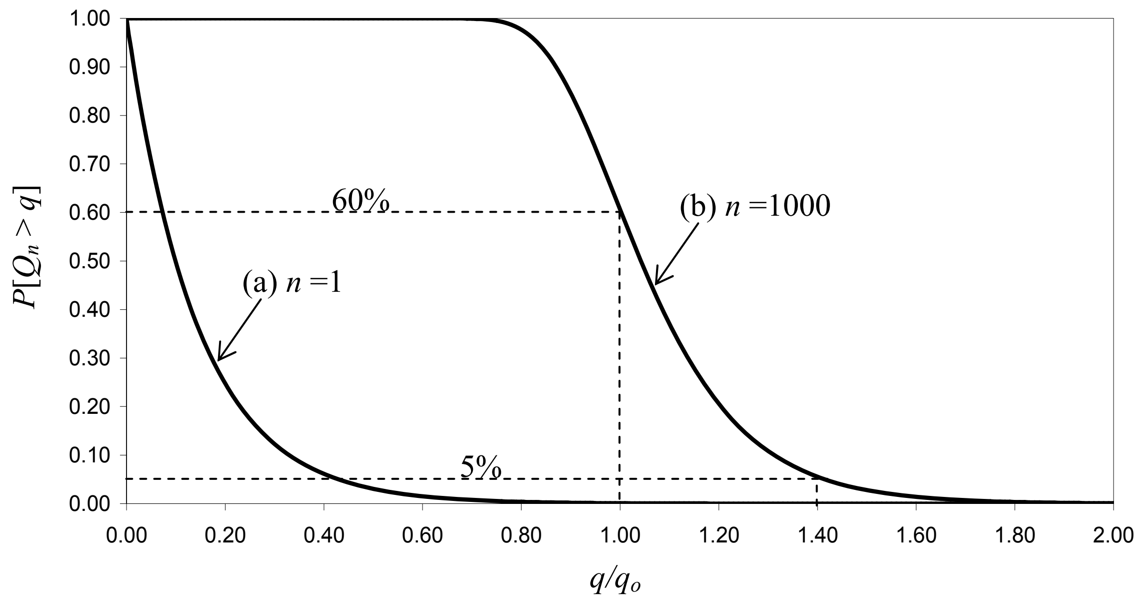

Figure 2: Exceedance probability $P[Q_n > q]$ of the maximum load Q_n from n waves as a function of q/q_o, where q_o is the design load. The curves correspond to $C = 2$ and design wave height h_o=1.868H_s.

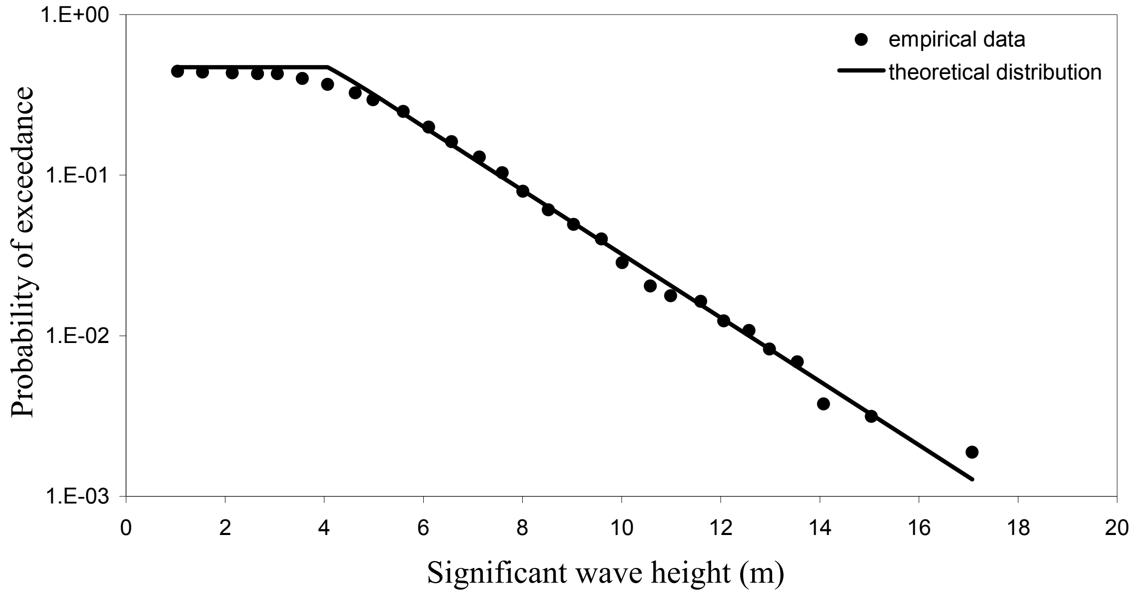

Figure 3: Annual probability of exceedance of significant wave height based on 1600 years of simulation.