Wigner-Eckart Theorem

CTDL, pages 999 - 1085, esp. 1048-1053

Last lecture on $1e^-$ Angular Part
Next: 2 lectures on $1e^-$ radial part
Many-e^- problems

What do we know about 1 particle angular momentum?

1. $|JM\rangle$ Basis set

   $[\mathbf{J}_i, \mathbf{J}_j] = i\hbar \sum_k \varepsilon_{ijk} \mathbf{J}_k$ definition $\rightarrow$ all matrix elements in $|JM_J\rangle$ basis set.

2. $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ Coupling of 2 angular momenta

   coupled $\leftrightarrow$ uncoupled basis sets

   transformation via $\mathbf{J}_- = \mathbf{L}_- + \mathbf{S}_-$ plus orthogonality. Also more general methods.

   $\mathbf{H}^{SO} + \mathbf{H}^{Zeeman}$ example
   * easy vs. hard basis sets
   * limiting cases, correlation diagram
   * pert. theory – patterns at both limits plus distortion

TODAY:

1. Define Scalar, Vector, Tensor Operators via Commutation Rules. Classification of an operator tells us how it transforms under coordinate rotation.
2. Statement of the Wigner-Eckart Theorem
3. Derive some matrix elements from Commutation Rules

Scalar	S	$\Delta J = \Delta M = 0$, M independent
Vector	V	$\Delta J = 0, \pm 1$, $\Delta M = 0, \pm 1$, explicit M dependences of matrix elements

These commutation rule derivations of matrix elements are tedious. There is a more direct but abstract derivation via rotation matrices. The goal here is to learn how to use 3-j coefficients.

Classification of Operators via Commutation Rules with CLASSIFYING ANGULAR MOMENTUM

			ω	Like	components(μ)
scalar	"constant"		0	J = 0	$\mu = 0$
vector	3 components		1	J = 1	$\mu = 0 \leftrightarrow z$ $+1 \leftrightarrow -2^{1/2}(x+iy)$ $-1 \leftrightarrow +2^{-1/2}(x-iy)$
tensor	$(2\omega + 1)$ components	2nd	2	2	+2, …, −2
	[ω is "rank"]	3rd	3	3	

Spherical Tensor Components [CTDL, page 1089 #8] …

Definition: $\left[\mathbf{J}_{\pm}, \mathbf{T}_{\mu}^{(\omega)}\right] = \hbar\left[\omega(\omega+1) - \mu(\mu \pm 1)\right]^{1/2} \mathbf{T}_{\mu\pm1}^{(\omega)}$

$$\left[\mathbf{J}_{z}, \mathbf{T}_{\mu}^{(\omega)}\right] = \hbar\mu \mathbf{T}_{\mu}^{(\omega)}$$

This classification is useful for matrix elements of $\mathbf{T}_{\mu}^{(\omega)}$ in $|JM\rangle$ basis set.

Example: $\mathbf{J} = \mathbf{L} + \mathbf{S}$

1. $[\vec{\mathbf{L}}, \vec{\mathbf{S}}] = 0 \quad \therefore$ **L** & **S** act as scalar operators with respect to each other.

common sense?
(vector analysis)

2. $\vec{\mathbf{L}}$ and $\vec{\mathbf{S}}$ act as vectors wrt **J**
3. $\vec{\mathbf{L}} \cdot \vec{\mathbf{S}}$ acts as scalar wrt **J**
4. $\vec{\mathbf{L}} \times \vec{\mathbf{S}}$ gives components of a vector wrt **J**.

[Because **L** × **S** is composed of products of components of two vectors, it could act as a second rank tensor. But it does not !]

[Nonlecture: given 1 and 2, prove 3]

Once operators are classified (classifications of same operator are different wrt different angular momenta), Wigner-Eckart Theorem provides angular factor of <u>all</u> matrix elements in any basis set!

$$\left\langle N'J'M' \middle| \mathbf{T}_{\mu}^{(\omega)} \middle| NJM \right\rangle = \underbrace{A_{M\mu[M']}^{J\omega J'}}_{\text{vector coupling coefficient}} \delta_{M',M+\mu} \underbrace{\left\langle N'J' \middle\| \mathbf{T}^{(\omega)} \middle\| NJ \right\rangle}_{\substack{\text{reduced matrix element} \\ \text{no } M', M, \text{ no } \mu}}$$

specifies everything else

redundant- usually omitted

rank of tensor – like an angular momentum

* triangle rule $|J-\omega| \le J' \le J+\omega$: selection rule for $T_\mu^{(\omega)}$, $\Delta J = \pm\omega, \pm(\omega-1), \ldots 0.$

* reduced matrix element contains all radial dependence – when there is no radial factor in the operator, then the J',J dependence can often be evaluated as well

* vector coupling coefficients are tabulated – lots of convenient symmetry properties: A.R. Edmonds, "Ang. Mom. in Q.M.", Princeton Univ. Press (1974).

Nonlecture: **L** & **S** act as vectors wrt **J** but scalars wrt each other

- $[\mathbf{L},\mathbf{S}] = 0$ scalars wrt each other
- $[\mathbf{J},\mathbf{L}] = [\mathbf{L}+\mathbf{S},\mathbf{L}] = [\mathbf{L},\mathbf{L}]$ $\therefore$ vector wrt. **J** if **L** is an angular momentum

  components of **L** satisfy the $\mathbf{T}_\mu^{(1)}$ definition

  $$\boxed{\mathbf{T}_{+1}^{(1)}[L]} = -2^{-1/2}\left[\mathbf{L}_x + i\mathbf{L}_y\right]$$

- $$\left[\mathbf{J}_z, -2^{-1/2}\left[\mathbf{L}_x + i\mathbf{L}_y\right]\right] = -2^{-1/2} i\hbar\left[\mathbf{L}_y - i\mathbf{L}_x\right] = -2^{-1/2}\hbar\left[\mathbf{L}_x + i\mathbf{L}_y\right]$$
  $$= +\hbar\mathbf{T}_{+1}^{(1)}[L]$$
  $$\left[\mathbf{J}_z, \mathbf{T}_{+1}^{(1)}[L]\right] = \hbar(+1)\mathbf{T}_{+1}^{(1)}[L]$$

This notation means: construct an operator classified as $T_{+1}^{(1)}$ out of components of $\vec{\mathbf{L}}$.

- $[\mathbf{J},\mathbf{S}] = [\mathbf{S},\mathbf{S}] \therefore$ **S** is vector wrt **J**

  $\mathbf{J} = \mathbf{L} + \mathbf{S}$

  etc.

• • • Show that $\mathbf{L}\cdot\mathbf{S}$ acts as scalar wrt **J**

$$\left[\mathbf{J}_\pm, \mathbf{L}\cdot\mathbf{S}\right] = \left[\mathbf{J}_x, \mathbf{L}_x\mathbf{S}_x + \mathbf{L}_y\mathbf{S}_y + \mathbf{L}_z\mathbf{S}_z\right] \pm i\left[\mathbf{J}_y, \mathbf{L}_x\mathbf{S}_x + \mathbf{L}_y\mathbf{S}_y + \mathbf{L}_z\mathbf{S}_z\right]$$
$$= \text{four terms}$$

$$\begin{aligned}
\left[\mathbf{J}_\pm, \mathbf{L}\cdot\mathbf{S}\right] &= \left[\mathbf{L}_x, \mathbf{L}_x\mathbf{S}_x + \mathbf{L}_y\mathbf{S}_y + \mathbf{L}_z\mathbf{S}_z\right] + \left[\mathbf{S}_x, \mathbf{L}_x\mathbf{S}_x + \mathbf{L}_y\mathbf{S}_y + \mathbf{L}_z\mathbf{S}_z\right] \\
&\quad \pm i\left[\mathbf{L}_y, \mathbf{L}_x\mathbf{S}_x + \mathbf{L}_y\mathbf{S}_y + \mathbf{L}_z\mathbf{S}_z\right] \pm i\left[\mathbf{S}_y, \mathbf{L}_x\mathbf{S}_x + \mathbf{L}_y\mathbf{S}_y + \mathbf{L}_z\mathbf{S}_z\right] \\
&= \Big[i\hbar\left(\mathbf{L}_z\mathbf{S}_y - \mathbf{L}_y\mathbf{S}_z\right) + i\hbar\left(\mathbf{L}_y\mathbf{S}_z - \mathbf{L}_z\mathbf{S}_y\right) \\
&\quad \pm ii\hbar\left(-\mathbf{L}_z\mathbf{S}_x - \mathbf{L}_x\mathbf{S}_z\right) \pm ii\hbar\left(-\mathbf{L}_x\mathbf{S}_z + \mathbf{L}_z\mathbf{S}_x\right)\Big] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\left[\mathbf{J}_z, \mathbf{L}\cdot\mathbf{S}\right] &= \left[\mathbf{L}_z, \mathbf{L}_x\mathbf{S}_x + \mathbf{L}_y\mathbf{S}_y + \mathbf{L}_z\mathbf{S}_z\right] + \left[\mathbf{S}_z, \mathbf{L}_x\mathbf{S}_x + \mathbf{L}_y\mathbf{S}_y + \mathbf{L}_z\mathbf{S}_z\right] \\
&= i\hbar\left(\mathbf{L}_y\mathbf{S}_x - \mathbf{L}_x\mathbf{S}_y\right) + i\hbar\left(\mathbf{L}_x\mathbf{S}_y - \mathbf{L}_y\mathbf{S}_x\right) \\
&= 0
\end{aligned}$$

This notation means: construct an operator classified as $T_0^{(0)}$ out of the components of **A** and **B.**

$$\mathbf{T}_0^{(0)}[A,B] = \sum_{k=-\omega}^{\omega} (-1)^k \mathbf{T}_k^{(\omega)}[A]\mathbf{T}_{-k}^{(\omega)}[B]$$

$\therefore \mathbf{L}\cdot\mathbf{S}$ acts as $\mathbf{T}_0^{(0)}$

$$\text{all serve same function}\begin{cases}\text{Vector Coupling Coefficients}\\ \text{Clebsch - Gordan Coefficients}\\ \text{3 - J coefficients}\\ \text{all related to what you already know how to obtain by ladders and orthogonality for}\end{cases}$$

$$|JJ_1J_2M\rangle = \sum_{\substack{M_2=M-M_1\\ M_2=-J_2}}^{+J_2} |J_1M_1J_2M_2\rangle \underbrace{\langle J_1M_1J_2M_2|JJ_1J_2M\rangle}_{\text{v.c. coefficient}} \qquad \text{completeness}$$

p. 46 Edmonds (1974) general formula

$$3-\text{J:}\quad \begin{pmatrix} J_1 & J_2 & J_3 \\ M_1 & M_2 & M_3 \end{pmatrix} = (-1)^{J_1-J_2-M_3}(2J_3+1)^{-\frac{1}{2}}\left(J_1M_1J_2M_2|J_1J_2J_3-M_3\right)$$

Constraint: $M_1+M_2+M_3=0$ This constraint is imposed in (|) notation but not in ⟨ | ⟩ notation.]

W-E Theorem is an extension of V-C idea because we think of operators as "like angular momenta" and we couple them to angular momenta to make new angular momentum eigenstates.

What is so great about W-E Theorem?

vast reduction of independent matrix elements
e.g. J = 10, ω = 1 (vector operator)
possible values of J′ limited to 9, 10, 11 by triangle rule

$$\left\langle J'M'\middle|\mathbf{T}_{\mu}^{(1)}\middle|JM\right\rangle$$

	Total # of M. E.		#R.M.E.	
J′ = 9	(2•9+1)(2•10+1)	399	1	$\langle 9\|\mathbf{T}_{\mu}^{(1)}\|10\rangle$
10	(2•10+1)(2•10+1)	441	1	$\langle 10\|\mathbf{T}_{\mu}^{(1)}\|10\rangle$
11	(2•11+1)(2•10+1)	483	1	$\langle 11\|\mathbf{T}_{\mu}^{(1)}\|10\rangle$
		1323	3	

CTD-L, pages 1048-1053

Outline proof of various parts of W-E Theorem

Scalar Operators **S**

$[\mathbf{J}_i,\mathbf{S}] = 0$ Definition (for all i)

1. $\Delta J = 0$ selection rule from $\left[\mathbf{J}^2,\mathbf{S}\right] = 0$
2. $\Delta M = 0$ selection rule from $\left[\mathbf{J}_z,\mathbf{S}\right] = 0$
3. M - independence from $\left[\mathbf{J}_\pm,\mathbf{S}\right] = 0$

1. show $\Delta J = 0$: $\langle J'M|\mathbf{S}|JM\rangle = 0$ if $J' \neq J$

$$\left[\mathbf{J}^2,\mathbf{S}\right] = 0$$

$$0 = \left\langle J'M'\middle|\left(\mathbf{J}^2\mathbf{S} - \mathbf{S}\mathbf{J}^2\right)\middle|JM\right\rangle = \hbar^2\left[J'(J'+1) - J(J+1)\right]\left\langle J'M'\middle|\mathbf{S}\middle|JM\right\rangle$$

direction of operation by $\mathbf{J}^2$

either $J' = J$ or $\left\langle J'M'\middle|\mathbf{S}\middle|JM\right\rangle = 0$

(only $\Delta J = 0$ matrix elements of **S** can be nonzero)

2. show $\Delta M = 0$: $\left\langle JM'\middle|\mathbf{S}\middle|JM\right\rangle = 0$ if $M' \neq M$

$$\left[\mathbf{J}_z,\mathbf{S}\right] = 0$$

$$0 = \left\langle JM'\middle|\left(\mathbf{J}_z\mathbf{S} - \mathbf{S}\mathbf{J}_z\right)\middle|JM\right\rangle = \hbar(M' - M)\left\langle JM'\middle|\mathbf{S}\middle|JM\right\rangle$$

either $M' = M$ or $\left\langle JM'\middle|\mathbf{S}\middle|JM\right\rangle = 0$

3 show M - Independence of matrix elements

uses $\Delta J = \Delta M = 0$ for **S**

$$\left[\mathbf{J}_\pm,\mathbf{S}\right] = 0$$

direction of operation by **S**

$$\begin{aligned} 0 = \left\langle JM'\middle|\left(\mathbf{J}_\pm\mathbf{S} - \mathbf{S}\mathbf{J}_\pm\right)\middle|JM\right\rangle &= \mathbf{s}_{JM}\left\langle JM'\middle|\mathbf{J}_\pm\middle|JM\right\rangle \\ &\quad - \mathbf{s}_{JM'}\left\langle JM'\middle|\mathbf{J}_\pm\middle|JM\right\rangle \\ &= \left(\mathbf{s}_{JM} - \mathbf{s}_{JM'}\right)\left\langle JM'\middle|\mathbf{J}_\pm\middle|JM\right\rangle \end{aligned}$$

we already know that **S** is diagonal in M.

[Should skip pages 27-6,7,8 and go directly to recursion relationship on page 27-10.]

Thus either $\mathbf{s}_{JM} = \mathbf{s}_{JM'}$ or $\left\langle JM' \middle| \mathbf{J}_{\pm} \middle| JM \right\rangle = 0$

Thus $\mathbf{s}_{JM}$ is independent of M

Putting all results for $\mathbf{S}$ together

$$\left\langle J'M' \middle| \mathbf{S} \middle| JM \right\rangle = \delta_{J'J}\delta_{M'M}\left\langle J \| \mathbf{S} \| J \right\rangle$$

Vector Operators $\mathbf{V}$

$$\left[\mathbf{J}_i, \mathbf{V}_j\right] = i\hbar \sum_k \varepsilon_{ijk}\mathbf{V}_k$$

1. M selection rules from $\left[\mathbf{J}_z, \vec{\mathbf{V}}\right]$
2. J selection rules from $\left[\mathbf{J}^2, \left[\mathbf{J}^2, \vec{\mathbf{V}}\right]\right]$
3. M - dependence of matrix elements of $\mathbf{V}$ from double commutator

1. <u>M selection rules</u>

a. $\left[\mathbf{J}_z, \mathbf{V}_z\right] = 0$

$$0 = \left\langle J'M' \middle| \left(\mathbf{J}_z\mathbf{V}_z - \mathbf{V}_z\mathbf{J}_z\right) \middle| JM \right\rangle = \hbar\left(M' - M\right)\left\langle J'M' \middle| \mathbf{V}_z \middle| JM \right\rangle$$

either $M = M'$ or $ME = 0$

b.
$$\left[\mathbf{J}_z, \mathbf{V}_{\pm}\right] = \left[\mathbf{J}_z, \mathbf{V}_x\right] \pm i\left[\mathbf{J}_z, \mathbf{V}_y\right] = i\hbar\left(\mathbf{V}_y \pm i\left(-\mathbf{V}_x\right)\right)$$
$$= \pm\hbar\mathbf{V}_{\pm}$$
$$\left\langle J'M' \middle| \left(\mathbf{J}_z\mathbf{V}_{\pm} - \mathbf{V}_{\pm}\mathbf{J}_z\right) \middle| JM \right\rangle = \pm\hbar\left\langle J'M' \middle| \mathbf{V}_{\pm} \middle| JM \right\rangle$$
$$\hbar\left(M' - M\right)\left\langle J'M' \middle| \mathbf{V}_{\pm} \middle| JM \right\rangle = \pm\hbar\left\langle J'M' \middle| \mathbf{V}_{\pm} \middle| JM \right\rangle$$
$$\hbar\left(M' - M \mp 1\right)\left\langle J'M' \middle| \mathbf{V}_{\pm} \middle| JM \right\rangle = 0$$
$$M' = M \pm 1 \quad \text{or} \quad ME = 0$$

Thus we have selection rules: $\mathbf{V}_z$ acts like $\mathbf{J}_z$ on M
$\mathbf{V}_{\pm}$ acts like $\mathbf{J}_{\pm}$ on M

2. M selection rules

need to use a result that requires lengthy derivation

$$\left[\mathbf{J}^2,\left[\mathbf{J}^2,\mathbf{V}\right]\right] = 2\hbar^2\left[\mathbf{J}^2\mathbf{V} - 2(\mathbf{J}\cdot\mathbf{V})\mathbf{J} + \mathbf{V}\mathbf{J}^2\right]$$

see proof in Condon and Shortley, pages 59-60

Take $\langle J'M'| \quad |JM\rangle$ Matrix elements of both sides of above Eq.

$$LHS = \left\langle J'M'\middle|\mathbf{J}^2(\mathbf{J}^2\mathbf{V}) - \mathbf{J}^2\mathbf{V}\mathbf{J}^2 - \mathbf{J}^2\mathbf{V}\mathbf{J}^2 + \mathbf{V}\mathbf{J}^2\mathbf{J}^2\middle|JM\right\rangle$$

$$= \hbar^4\left[\left(J'(J'+1)\right)^2 - 2J(J+1)J'(J'+1) + J^2(J+1)^2\right]\left\langle J'M'\middle|\vec{\mathbf{V}}\middle|JM\right\rangle$$

$$RHS = 2\hbar^4\left[J'(J'+1) + J(J+1)\right]\left\langle J'M'\middle|\vec{\mathbf{V}}\middle|JM\right\rangle - 4\hbar^4\left\langle J'M'\middle|\underbrace{(\mathbf{J}\cdot\mathbf{V})}_{\text{scalar}}\mathbf{J}\middle|JM\right\rangle$$

$$\left\langle J'M'\middle|(\mathbf{J}\cdot\mathbf{V})\mathbf{J}\middle|JM\right\rangle = \sum_{J''M''}\left\langle J'M'\middle|(\mathbf{J}\cdot\mathbf{V})\middle|_{J'=J'',M'=M''} J''M''\right\rangle\left\langle J''M''\middle|\mathbf{J}\middle|JM\right\rangle$$

$$= \left\langle J'\middle\|\mathbf{J}\cdot\mathbf{V}\middle\|J'\right\rangle\underbrace{\left\langle J'M'\middle|\mathbf{J}\middle|JM\right\rangle}_{J'=J}$$

$$= \left\langle J\middle\|\mathbf{J}\cdot\mathbf{V}\middle\|J\right\rangle\left\langle JM'\middle|\mathbf{J}\middle|JM\right\rangle\delta_{J'J}$$

two cases for overall matrix element

A. $J' \neq J$

B. $J' = J$

A. $J' \neq J$

$$RHS = 2\hbar^4\left[J'(J'+1) + J(J+1)\right]\left\langle J'M'\middle|\vec{\mathbf{V}}\middle|JM\right\rangle$$

$$LHS = \hbar^4\left[J'^2(J'+1)^2 - 2J(J+1)J'(J'+1) + J^2(J+1)^2\right]\left\langle J'M'\middle|\mathbf{V}\middle|JM\right\rangle$$

$$0 = LHS - RHS = \text{algebra} = \hbar^4\langle J'M'|\mathbf{V}|JM\rangle\left[(J'-J)^2 - 1\right]\left[(J'+J+1)^2 - 1\right]$$

ME = 0 unless J′ = J ± 1 or J′ = –J

(J ′ = –J is impossible except for J′ = –J = 0, but this violates J′ ≠ J assumption)

$\therefore \Delta J = \pm 1$ selection rule for $\vec{\mathbf{V}}$

B. $J' = J$

$$LHS = 0$$

$$0 = RHS = 4\hbar^2\left[\hbar^2 J(J+1)\left\langle JM'\middle|\mathbf{V}\middle|JM\right\rangle - \left\langle J\middle\|\mathbf{J}\cdot\mathbf{V}\middle\|J\right\rangle\left\langle JM'\middle|\mathbf{J}\middle|JM\right\rangle\right]$$

$$\left\langle JM'\middle|\vec{\mathbf{V}}\middle|JM\right\rangle = \underbrace{\frac{\left\langle J\middle\|\mathbf{J}\cdot\mathbf{V}\middle\|J\right\rangle}{\hbar^2 J(J+1)}}_{C_0(J)}\left\langle JM'\middle|\vec{\mathbf{J}}\middle|JM\right\rangle$$

A WONDERFUL AND MEMORABLE RESULT. It says that all $\Delta J = 0$ matrix elements of $\vec{\mathbf{V}}$ are $\propto$ corresponding matrix element of $\vec{\mathbf{J}}$! A simplified form of W - E Theorem for vector operators.

Lots of (NONLECTURE) algebra needed to generate all $\Delta J = \pm 1$ matrix elements of $\vec{V}$.

SUMMARY OF C.R. RESULTS: Wigner-Eckart Theorem for Vector Operator

$$\Delta J = 0 \quad \langle JM|\mathbf{V}_z|JM\rangle = C_o(J)M$$

$$\langle JM \pm 1|\mathbf{V}_\pm|JM\rangle = C_o(J)\left[J(J+1) - M(M\pm 1)\right]^{1/2}$$

special case: These are exactly the same form as corresponding matrix element of $\mathbf{J}_i$.

$$\Delta J = +1 \qquad \langle J+1M\pm 1|\mathbf{V}_\pm|JM\rangle = \mp C_+(J)\left[(J\pm M+2)(J\pm M+1)\right]^{1/2}$$

$$\langle J+1M|\mathbf{V}_z|JM\rangle = +C_+(J)\left[(J+M+1)(J-M+1)\right]^{1/2}$$

$$\Delta J = -1 \qquad \langle J-1M\pm 1|\mathbf{V}_\pm|JM\rangle = \pm C_-(J)\left[(J\mp M)(J\pm M+1)\right]^{1/2}$$

$$\langle J-1M|\mathbf{V}_z|JM\rangle = +C_-(J)\left[(J-M)(J+M)\right]^{1/2}$$

only $C_o(J)$, $C_+(J)$, $C_-(J)$: 3 unknown J-dependent constants for each J.

NONLECTURE (to end of notes). Example of how recursion relationships (reduced matrix elements) are derived for each possible ΔJ.

$\Delta J = \pm 1$ matrix elements of $\vec{\mathbf{V}}$

$\Delta M = +1$ using $\left[\mathbf{J}_+, \mathbf{V}_+\right] = 0$

ΔM selection rule for $\mathbf{V}_+$ is $\Delta M = +1$

ΔM selection rule for $\mathbf{J}_+\mathbf{V}_+$ is $\Delta M = +2$

$$\langle|CR|\rangle = 0 = \langle J+1M+1|(\mathbf{J}_+\mathbf{V}_+ - \mathbf{V}_+\mathbf{J}_+)|JM-1\rangle$$

(arrow denotes $\mathbf{J}_+$ operates to right)

$$0 = \langle J+1M+1|\mathbf{J}_+|J+1M\rangle\langle J+1M|\mathbf{V}_+|JM-1\rangle$$

$$-\langle J+1M+1|\mathbf{V}_+|JM\rangle\langle JM|\mathbf{J}_+|JM-1\rangle$$

expand using completeness

($\mathbf{J}_+$ operates to left)

$$\frac{\langle J+1M|\mathbf{V}_+|JM-1\rangle}{\langle JM|\mathbf{J}_+|JM-1\rangle} = \frac{\langle J+1M+1|\mathbf{V}_+|JM\rangle}{\langle J+1M+1|\mathbf{J}_+|J+1M\rangle}$$

$$\hbar\left[(J+M)\cancel{(J-M+1)}\right]^{1/2} \qquad \hbar\left[(J+M+2)\cancel{(J-M+1)}\right]^{1/2}$$

The matrix elements in the denominator are to be replaced by their values, and a common factor is cancelled multiply both sides by $[J+M+1]^{-1/2}$ to display symmetry

$$\frac{\langle J+1M|\mathbf{V}_+|JM-1\rangle}{(J+M)^{1/2}(J+M+1)^{1/2}} = -\frac{\langle J+1M+1|\mathbf{V}_+|JM\rangle}{(J+M+1)^{1/2}(J+M+2)^{1/2}} \equiv -C_+(J)$$

$M \to M+1$ recursion relationship

sign chosen so that $\mathbf{V}_z$ matrix elements will be $+C_+(J)$

ratio is independent of M

$$C_+(J) \equiv \langle \alpha' J+1 \| \mathbf{V} \| \alpha J \rangle$$

$$\langle J+1M+1|\mathbf{V}_+|JM\rangle = -C_+[(J+M+1)(J+M+2)]^{1/2}$$

Remaining to do for $\Delta J = \pm 1$ matrix elements

A. $[\mathbf{J}_-, \mathbf{V}_+] = -2\hbar\mathbf{V}_z$ gives $\mathbf{V}_z$ matrix element when we take $\Delta M = 0$ matrix element of both sides

B. $[\mathbf{J}_-, \mathbf{V}_z] = \hbar\mathbf{V}_-$ gives $\mathbf{V}_-$ matrix element when we take $\Delta M = -1$ matrix element of both sides

A. $[\mathbf{J}_-, \mathbf{V}_+] = -2\hbar\mathbf{V}_z$ $\Delta M = 0$ selection rule for both sides

$$RHS = -2\hbar\langle J+1M|\mathbf{V}_z|JM\rangle$$

$$\begin{aligned}
LHS &= \langle J+1M|(\mathbf{J}_-\mathbf{V}_+ - \mathbf{V}_+\mathbf{J}_-)|JM\rangle \\
&= \langle J+1M|\mathbf{J}_-|J+1M+1\rangle\langle J+1M+1|\mathbf{V}_+|JM\rangle \\
&\quad -\langle J+1M|\mathbf{V}_+|JM-1\rangle\langle JM-1|\mathbf{J}_-|JM\rangle \\
&= \hbar[(J+1)(J+2)-M(M+1)]^{1/2}\langle J+1M+1|\mathbf{V}_+|JM\rangle \\
&\quad -\hbar[J(J+1)-M(M-1)]^{1/2}\langle J+1M|\mathbf{V}_+|JM-1\rangle
\end{aligned}$$

rearrange this and use $\mathbf{V}_+$ recursion rule from above

$$\begin{aligned}
LHS &= \hbar C_+(J)[(J+M+1)(J-M+1)]^{1/2}[(J+M)-(J+M+2)]^1 \\
&= -2\hbar C_+(J)[(J+M+1)(J-M+1)]^{1/2}
\end{aligned}$$

RHS = LHS

$$\left\langle J+1M\middle|\mathbf{V}_z\middle|JM\right\rangle = C_+(J)\left[(J+M+1)(J-M+1)\right]^{1/2}$$

B. $\left[\mathbf{J}_-,\mathbf{V}_z\right] = \hbar\mathbf{V}_-$ take $\left\langle J+1M-1\middle|\cdots\middle|JM\right\rangle$

$$RHS = \hbar\left\langle J+1M-1\middle|\mathbf{V}_-\middle|JM\right\rangle$$

$$LHS = \left\langle J+1M-1\middle|\mathbf{J}_-\middle|J+1M\right\rangle\left\langle J+1M\middle|\mathbf{V}_z\middle|JM\right\rangle$$

$$-\left\langle J+1M-1\middle|\mathbf{V}_z\middle|JM-1\right\rangle\left\langle JM-1\middle|\mathbf{J}_-\middle|JM\right\rangle$$

$$= \hbar C_+(J)\left[(J-M+2)(J-M+1)\right]^{1/2}$$

$$\left\langle J+1M-1\middle|\mathbf{V}_-\middle|JM\right\rangle = +C_+(J)\left[(J-M+2)(J-M+1)\right]^{1/2}$$

VERY COMPLICATED AND TEDIOUS ALGEBRA