3D-Central Force Problems II

Last time: $[\mathbf{x},\mathbf{p}] = i\hbar \rightarrow$ vector commutation rules: generalize from 1-D to 3-D conjugate position and momentum components in Cartesian coordinates

Correspondence Principle Recipe
- Cartesian and vector analysis
- Symmetrize (make it Hermitian)
- classical in $\hbar \rightarrow 0$ limit

Derived key results:

$$\left[f(\mathbf{x}),\mathbf{p_x}\right] = i\hbar\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$$

$$\left[f(\mathbf{r}),\mathbf{q}\cdot\mathbf{p}\right] = i\hbar\frac{\partial \mathbf{f}}{\partial \mathbf{r}}\mathbf{r} \quad \text{based on } \frac{d\mathbf{f}}{d\mathbf{r}}\frac{\partial \mathbf{r}}{\partial \mathbf{x}} \text{ and } \mathbf{r} = \left[\mathbf{x}^2+\mathbf{y}^2+\mathbf{z}^2\right]^{1/2}$$

$^*\mathbf{p_r} = \mathbf{r}^{-1}(\mathbf{q}\cdot\mathbf{p} - i\hbar) \leftarrow$ (came from symmetrization in Cartesian coordinates)

$^*\mathbf{p}^2 = \mathbf{p_r}^2 + \mathbf{r}^{-2}\mathbf{L}^2$ operator algebra gave simple separation of variables

$^*\mathbf{L} = \mathbf{q}\times\mathbf{p}$ not necessary (or possible) to symmetrize

$^*\mathbf{H} = \frac{\mathbf{p_r}^2}{2\mu} + \left[\frac{\mathbf{L}^2}{2\mu\mathbf{r}^2} + V(\mathbf{r})\right]$ $V_\ell(r)$ radial effective potential

We do not yet know anything about $\mathbf{L}^2$ nd $\mathbf{L}_i$.

TODAY [purpose is mostly to practice [,] and angular momentum algebra]

Obtain angular Momentum Commutation Rules $\rightarrow$ Block diagonalize **H**

ε_{ijk} Levi-Civita Antisymmetric Tensor

useful in derivations, vector commutators, and remembering stuff.

Next Lecture: Begin derivation of all angular momentum matrix elements starting from Commutation Rule definitions of angular momentum.

GOALS

1. $\left[\mathbf{L}_i, f(\mathbf{r})\right] = 0$ any scalar function of scalar **r**.
2. $\left[\mathbf{L}_i, \mathbf{p_r}\right] = 0$ difficult - need ε_{ijk}
3. $\left[\mathbf{L}_i, \mathbf{p}_\mathbf{r}^2\right] = 0$
4. $\left[\mathbf{L}_i, \mathbf{L}^2\right] = 0$ $\left(\text{but } \left[\mathbf{L}_i, \mathbf{L}_j^2\right] \neq 0!\right)$
5. C.S.C.O. $\mathbf{H}, \mathbf{L}^2, \mathbf{L}_i \rightarrow$ block diagonalize **H**

These 1-4 are chosen to show that all terms in **H** commute with $\mathbf{L}^2$ and $\mathbf{L}_i$

1. $$\left[\mathbf{L}_z, f(\mathbf{r})\right] = \left[\mathbf{x}\mathbf{p}_y - \mathbf{y}\mathbf{p}_x, f(\mathbf{r})\right] = \mathbf{x}\left[\mathbf{p}_y, f\right] + [\mathbf{x}, f]\mathbf{p}_y - \mathbf{y}\left[\mathbf{p}_x, f\right] - [\mathbf{y}, f]\mathbf{p}_x$$

$[\mathbf{x}, f] = 0,\qquad [\mathbf{y}, f] = 0$ because $\left[\vec{\mathbf{q}}, f(\mathbf{r})\right] = 0\hat{i} + 0\hat{j} + 0\hat{k}$

$$\text{recall } \left[f(\mathbf{r}), \mathbf{p}_x\right] = i\hbar \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} = i\hbar \frac{\partial f}{\partial r}\frac{x}{r}$$

$$\left[\mathbf{L}_z, f(\mathbf{r})\right] = -i\hbar \frac{\partial f}{\partial r}\left[x\frac{y}{r} - y\frac{x}{r}\right] = 0$$

2. $$\left[\mathbf{L}_z, \mathbf{p_r}\right] = \left[\mathbf{L}_z, \mathbf{r}^{-1}(\mathbf{q}\cdot\mathbf{p} - i\hbar)\right] = \left[\mathbf{L}_z, \mathbf{r}^{-1}\mathbf{q}\cdot\mathbf{p}\right]$$

$$= \underbrace{\left[\mathbf{L}_z, \mathbf{r}^{-1}\right]}_{0}\mathbf{q}\cdot\mathbf{p} + \mathbf{r}^{-1}\left[\mathbf{L}_z, \mathbf{q}\cdot\mathbf{p}\right]$$

$$\left[\mathbf{L}_z, \mathbf{q}\cdot\mathbf{p}\right] = \mathbf{q}\cdot\left[\mathbf{L}_z, \vec{\mathbf{p}}\right] + \left[\mathbf{L}_z, \vec{\mathbf{q}}\right]\cdot\mathbf{p}$$ two vector commutators on RHS

Note that $\vec{\mathbf{q}}$ is not $f(\mathbf{r})$!

need to define special notational trick to evaluate these DIFFICULT COMMUTATORS

Levi-Civita Symbol	ε_{ijk}
cyclic order	$\varepsilon_{xyz} = \varepsilon_{yzx} = \varepsilon_{zxy} = +1$
adjacent interchange	$\varepsilon_{yxz} = \varepsilon_{zyx} = \varepsilon_{xzy} = -1$
2 repeated indices	$\varepsilon_{xxy} =$ etc. $= 0$

I claim $[\mathbf{L}_i, \mathbf{p}_j] = i\hbar \sum_k \varepsilon_{ijk} \mathbf{p}_k$. This will become the *definition* of a "vector operator" with respect to **L**.

Nonlecture: Verify claim for 1 of 3 × 3 = 9 possible cases

let i = x, j = y

$$[\mathbf{L}_x, \mathbf{p}_y] = [\mathbf{y}\mathbf{p}_z - \mathbf{z}\mathbf{p}_y, \mathbf{p}_y] = [\mathbf{y}\mathbf{p}_z, \mathbf{p}_y] + 0$$

$$= [\mathbf{y}, \mathbf{p}_y]\mathbf{p}_z + \mathbf{y}\underbrace{[\mathbf{p}_z, \mathbf{p}_y]}_{0}$$

$$= i\hbar \mathbf{p}_z$$

Now check this using ε_{ijk}

$$[\mathbf{L}_x, \mathbf{p}_y] = i\hbar \sum_k \varepsilon_{xyk} \mathbf{p}_k = i\hbar[\underbrace{\varepsilon_{xyx}}_{0}\mathbf{p}_x + \underbrace{\varepsilon_{xyy}}_{0}\mathbf{p}_y + \varepsilon_{xyz}\mathbf{p}_z]$$

$$= i\hbar \mathbf{p}_z. \qquad \text{OK}$$

All other 8 cases go similarly

Other important Commutation Rules

$$[\mathbf{L_i}, \mathbf{p_j}] = i\hbar \sum_k \varepsilon_{ijk} \mathbf{p}_k$$

$$[\mathbf{L_i}, \mathbf{q_j}] = i\hbar \sum_k \varepsilon_{ijk} \mathbf{q}_k$$

general definition of a "vector" operator

$$[\mathbf{L_i}, \mathbf{L_j}] = i\hbar \sum_k \varepsilon_{ijk} \mathbf{L}_k$$

general definition of an "angular momentum." Works even for spin where $\mathbf{q} \times \mathbf{p}$ definition is inapplicable

All angular momentum matrix elements will be derived from these commutation rules.

FOR THE READER: VERIFY ONE COMPONENT OF EACH OF THE THREE ABOVE COMMUTATORS

$\left[\mathbf{L}_i,\mathbf{L}_j\right]=i\hbar\sum_k \varepsilon_{ijk}\mathbf{L}_k$ is identical to

$$\mathbf{L}\times\mathbf{L}=i\hbar\mathbf{L}$$

(expect 0! because vector cross product $\vec{A}\times\vec{B}=|A||B|\sin\theta\hat{\mathbf{e}}_{AB}$)

$$\mathbf{L}\times\mathbf{L}=\begin{pmatrix}\hat{i} & \hat{j} & \hat{k}\\ \mathbf{L}_x & \mathbf{L}_y & \mathbf{L}_z\\ \mathbf{L}_x & \mathbf{L}_y & \mathbf{L}_z\end{pmatrix}=\hat{i}\left(\mathbf{L}_y\mathbf{L}_z-\mathbf{L}_z\mathbf{L}_y\right)+\hat{j}\left(\mathbf{L}_z\mathbf{L}_x-\mathbf{L}_x\mathbf{L}_z\right)+\hat{k}\left(\mathbf{L}_x\mathbf{L}_y-\mathbf{L}_y\mathbf{L}_x\right)$$

$$=i\hbar\left[\hat{i}\mathbf{L}_x+\hat{j}\mathbf{L}_y+\hat{k}\mathbf{L}_z\right]=i\hbar\mathbf{L}$$

This vector cross product definition of **L** is more general than $\mathbf{q}\times\mathbf{p}$ because there is no way to define spin in $\mathbf{q}\times\mathbf{p}$ form but $\mathbf{S}\times\mathbf{S}=i\hbar\mathbf{S}$ is quite meaningful.

Can one generalize that, if $\mathbf{L}\times\mathbf{L}=i\hbar\mathbf{L}$ (instead of 0), and the $[\mathbf{L}_i, \mathbf{L}_j]$ and $[\mathbf{L}_i, \mathbf{p}_j]$ commutation rules have similar forms, that $\mathbf{L}\times\mathbf{p}=i\hbar\mathbf{p}$? NO! Check for yourself!

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2. Continued.

$$\left[\mathbf{L}_z,\mathbf{p}_r\right]=\mathbf{r}^{-1}\mathbf{q}\cdot\left[\mathbf{L}_z,\vec{\mathbf{p}}\right]+\mathbf{r}^{-1}\left[\mathbf{L}_z,\vec{\mathbf{q}}\right]\cdot\mathbf{p}$$ vector commutators

$$\left[\mathbf{L}_i,\vec{\mathbf{p}}\right]=i\hbar\sum_k\left(\hat{i}\varepsilon_{ixk}+\hat{j}\varepsilon_{iyk}+\hat{k}\varepsilon_{izk}\right)\mathbf{p}_k$$

sum of 3 terms

$$\mathbf{q}\cdot\left[\mathbf{L}_i,\vec{\mathbf{p}}\right]=i\hbar\sum_k\left(\mathbf{x}\varepsilon_{ixk}+\mathbf{y}\varepsilon_{iyk}+\mathbf{z}\varepsilon_{izk}\right)\mathbf{p}_k$$ only one of these terms is nonzero

$$=i\hbar\sum_{j,k}\varepsilon_{ijk}\mathbf{q}_j\mathbf{p}_k \qquad (1)$$

and the other term $[\mathbf{L}_i, \vec{\mathbf{q}}] \cdot \vec{\mathbf{p}}$

$$[\mathbf{L}_i, \vec{\mathbf{q}}] = i\hbar \sum_k \left[\hat{i}\varepsilon_{ixk} + \hat{j}\varepsilon_{iyk} + \hat{k}\varepsilon_{izk}\right]\mathbf{q}_k$$

$$[\mathbf{L}_i, \vec{\mathbf{q}}] \cdot \mathbf{p} = i\hbar \sum_k \left[\varepsilon_{ixk}\mathbf{q}_k\mathbf{p}_x + \varepsilon_{iyk}\mathbf{q}_k\mathbf{p}_y + \varepsilon_{izk}\mathbf{q}_k\mathbf{p}_z\right]$$

$$= i\hbar \sum_{j,k} \varepsilon_{ijk}\mathbf{q}_k\mathbf{p}_j = i\hbar \sum_{k,j} \varepsilon_{ikj}\mathbf{q}_j\mathbf{p}_k \qquad \text{(k}\leftrightarrow\text{j labels permuted)}$$

$$= -i\hbar \sum_{k,j} \varepsilon_{ijk}\mathbf{q}_j\mathbf{p}_k \qquad (2)$$

└── switch order of j and k

putting Eqs. (1) and (2) together

$$\mathbf{q} \cdot [\mathbf{L}_i, \mathbf{p}] + [\mathbf{L}_i, \mathbf{q}] \cdot \mathbf{p} = i\hbar \sum_{j,k} \left[\varepsilon_{ijk}\mathbf{q}_j\mathbf{p}_k - \varepsilon_{ijk}\mathbf{q}_j\mathbf{p}_k\right] = 0!$$

Elegance and power of ε_{ijk} notation!

We have shown that:

* $[\mathbf{L}_i, \mathbf{p}_r] = 0$ for all i
* easy now to show $[\mathbf{L}_i, \mathbf{p}_r^2] = 0$

Finally $$[\mathbf{L}_i, \mathbf{L}^2] = \sum_j [\mathbf{L}_i, \mathbf{L}_j^2] = \sum_j \left(\mathbf{L}_j[\mathbf{L}_i, \mathbf{L}_j] + [\mathbf{L}_i, \mathbf{L}_j]\mathbf{L}_j\right]$$

$$= \sum_j \left[\mathbf{L}_j\left(i\hbar \sum_k \varepsilon_{ijk}\mathbf{L}_k\right) + \left(i\hbar \sum_k \varepsilon_{ijk}\mathbf{L}_k\right)\mathbf{L}_j\right]$$

same trick: permute j ↔ k indices in second term

$$\varepsilon_{ijk} = -\varepsilon_{ikj} \qquad -\left(i\hbar \sum_k \varepsilon_{ijk}\mathbf{L}_j\right)\mathbf{L}_k$$

$$= 0$$

But be careful: $$[\mathbf{L}_i, \mathbf{L}_j^2] = \mathbf{L}_j[\mathbf{L}_i, \mathbf{L}_j] + [\mathbf{L}_i, \mathbf{L}_j]\mathbf{L}_j = i\hbar\left(\mathbf{L}_j \sum_k \varepsilon_{ijk}\mathbf{L}_k + \sum_k \varepsilon_{ijk}\mathbf{L}_k\mathbf{L}_j\right)$$

because this is a sum only over k, so can't combine and cancel terms.

for i = x, j = y

$$\left[\mathbf{L}_x, \mathbf{L}_y^2\right] = i\hbar\left[\mathbf{L}_y\mathbf{L}_z + \mathbf{L}_z\mathbf{L}_y\right] \neq 0!$$

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so we have shown

$$\left[\mathbf{L}^2, \mathbf{L}_i\right] = 0$$

$$\left[\mathbf{L}^2, f(\mathbf{r})\right] = 0$$

$$\left[\mathbf{L}_i, f(\mathbf{r})\right] = 0$$

$$\left[\mathbf{L}^2, \mathbf{p}_r\right] = 0$$

$$\left[\mathbf{L}_i, \mathbf{p}_r\right] = 0$$

$\therefore$ $\mathbf{L}^2$, $\mathbf{L}_i$, **H** all commute — **C**omplete **S**et of **M**utually **C**ommuting **O**perators

(eigenfunction of $\mathbf{L}^2$ with eigenvalue $\hbar^2 L(L+1)$)

So what does this tell us about $\langle L|\mathbf{H}|L'\rangle = ?$

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BLOCK DIAGONALIZATION OF **H**!

Basis functions $\quad \psi = \chi(r)\left|L^2, L_z\right\rangle = \left|nLM_L\right\rangle$

- $\chi(r)$: radial, special
- $\left|L^2, L_z\right\rangle$: angular, universal
- M_L: eigenfunctions of $\mathbf{L}_z$
- L: eigenfunctions of $\mathbf{L}^2$
- n: which radial eigenfunction?

Next time I will show, starting from

$$\boxed{\left[\mathbf{L}_i, \mathbf{L}_j\right] = i\hbar\sum_k \varepsilon_{ijk}\mathbf{L}_k}$$

, that

* $\mathbf{L}^2\left|nLM_L\right\rangle = \hbar^2 L(L+1)\left|nLM_L\right\rangle \qquad L = 0, 1, \ldots$

* $\mathbf{L}_z\left|nLM_L\right\rangle = \hbar M_L\left|nLM_L\right\rangle \qquad M_L = -L, -L+1, \ldots +L$

also derive all $\mathbf{L}_x$ and $\mathbf{L}_y$ matrix elements in $\left|nLM_L\right\rangle$ basis set.