End Matrix Solution of H-O, a + a† Operators

1. starting from $\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}k\mathbf{x}^2$ and $[\mathbf{x},\mathbf{p}] = i\hbar$

2. showed
$$p_{nm} = \frac{m}{i\hbar} x_{nm}(E_m - E_n)$$
$$x_{nm} = \frac{i}{\hbar k} p_{nm}(E_m - E_n)$$
$$\therefore x_{nn} = 0, p_{nn} = 0 \text{ and } \begin{Bmatrix} x_{nm} \\ p_{nm} \end{Bmatrix} = 0 \text{ if } E_n = E_m$$

3. $x_{nm}^2 = -\frac{1}{km} p_{nm}^2$

$E_m - E_n = \pm i\hbar\omega \qquad \omega = (k/m)^{1/2}$

$\therefore$ the only non-zero **x** and **p** elements are between states whose E's differ by $\pm\hbar\omega$

4. combs of connected states, block diag. of **H**, **x**, **p**, $\mathbf{x}^2$, $\mathbf{p}^2$ $E_n^{(i)} = \hbar\omega n + \varepsilon_i$

5. lowest index must exist because lowest E must exist. Call this index 0

$$|x_{01}|^2 = \frac{\hbar}{2}(km)^{-1/2}$$
$$|p_{01}|^2 = \frac{\hbar}{2}(km)^{+1/2}$$

$$\boxed{\text{from phase choice} \quad x_{01} = +i(km)^{-1/2} p_{01}}$$

Today

6. Recursion Relationship $|x_{nn+1}|^2$ in terms of $|x_{nn-1}|^2$
$$\Downarrow$$
general matrix elements $|x_{nn+1}|^2, |p_{nn+1}|^2$

7. general **x** and **p** elements

8. only blocks correspond to $\varepsilon_i = \frac{1}{2}\hbar\omega$

Dimensionless **x, p, H** and **a** (annihilation) and **a**† (creation) operators phase ambiguity: we can specify absolute phase of **x** or **p** BUT NOT BOTH because that would affect value of [**x**,**p**]

BY CONVENTION:

matrix elements of **x** are REAL
p are IMAGINARY

try $x_{01} = +i(km)^{-1/2} p_{01}$ and plug this into
$x_{01}p_{01}^{*} - p_{01}x_{01}^{*} = i\hbar$

get
$$\boxed{\begin{aligned} |x_{01}|^2 &= \frac{\hbar}{2}(km)^{-1/2} \\ |p_{01}|^2 &= \frac{\hbar}{2}(km)^{+1/2} \end{aligned}}$$

$$\left[\begin{array}{l} \text{If we had chosen } x_{01} = -i(km)^{-1/2} p_{01} \text{ we would have} \\ \text{obtained } |x_{01}|^2 = -\frac{\hbar}{2}(km)^{1/2} \text{ which is impossible!} \end{array}\right]$$

check for self-consistency of seemingly arbitrary phase choices at every opportunity:
* Hermiticity
* $|\quad|^2 \geq 0$

6. Recursion Relation for $|x_{ii+1}|^2$

start again with gerand equation derived in #3 above using the phase choice that worked above

$$x_{nn+1} = i(km)^{-1/2} p_{nn+1}$$

going up

Hermiticity $\quad x_{n+1n}^{*} = i(km)^{-1/2} p_{n+1n}^{*}$

c.c. of both sides $\quad x_{n+1n} = -i(km)^{-1/2} p_{n+1n}$

going down

$$\therefore \quad x_{nn\pm1} = \pm i(km)^{-1/2} p_{nn\pm1}$$

now the arbitrary part of the phase ambiguity in the relationship between **x** and **p** is eliminated

Apply this to the general term in [**x**,**p**] ⇒ algebra

NONLECTURE : from terms in $[\mathbf{x},\mathbf{p}] = i\hbar$

$$x_{nn+1}p_{n+1n} = x_{nn+1}p^*_{nn+1} = x_{nn+1}\left(-\frac{(km)^{1/2}}{i}x^*_{nn+1}\right)$$
$$= \left|x_{nn+1}\right|^2\left(+i(km)^{1/2}\right)$$

$$-p_{nn+1}x_{n+1n} = -\left(\frac{(km)^{1/2}}{i}x_{nn+1}\right)\left(x^*_{nn+1}\right) = \left|x_{nn+1}\right|^2\left(+i(km)^{1/2}\right)$$

$$x_{nn-1}p_{n-1n} = x_{nn-1}p^*_{nn-1} = x_{nn-1}\left(+\frac{(km)^{1/2}}{i}x^*_{nn-1}\right)$$
$$= \left|x_{nn-1}\right|^2\left(-i(km)^{1/2}\right)$$

$$-p_{nn-1}x_{n-1n} = -\left(-\frac{(km)^{1/2}}{i}x_{nn-1}\right)\left(x^*_{nn-1}\right) = \left|x_{nn-1}\right|^2\left(-i(km)^{1/2}\right)$$

$$\therefore \quad i\hbar = 2i(km)^{1/2}\left[\left|x_{nn+1}\right|^2 - \left|x_{nn-1}\right|^2\right]$$

$$\left|x_{nn+1}\right|^2 = \frac{\hbar(km)^{-1/2}}{2} + \left|x_{nn-1}\right|^2$$ recursion relation

but $\left|x_{01}\right|^2 = \left|x_{10}\right|^2 = \frac{\hbar}{2}(km)^{-1/2}$

thus
$$\boxed{\begin{aligned}\left|x_{nn+1}\right|^2 &= (n+1)\frac{\hbar}{2}(km)^{-1/2}\\ \left|p_{nn+1}\right|^2 &= (n+1)\frac{\hbar}{2}(km)^{+1/2}\end{aligned}}$$
general result

7. Magnitudes and Phases for $x_{nn\pm1}$ and $p_{nn\pm1}$

verify phase consistency and Hermiticity for **x** and **p**

in #3 we derived $x_{nn\pm1} = \pm i(km)^{-1/2} p_{nn\pm1}$

one self-consistent set is

x real and positive

$$\begin{cases} x_{nn+1} = +(n+1)^{1/2}\left(\frac{\hbar}{2(km)^{1/2}}\right)^{1/2} = +x_{n+1n} \\ x_{nn-1} = +(n)^{1/2}\left(\frac{\hbar}{2(km)^{1/2}}\right)^{1/2} = +x_{nn-1} \end{cases}$$

$$(km)^{1/2} = m\omega$$

AND

p imaginary w/sign flip for up vs. down

$$\begin{cases} p_{nn+1} = -i(n+1)^{1/2}\left(\frac{\hbar(km)^{1/2}}{2}\right)^{1/2} = -p_{n+1n} \\ p_{nn-1} = +i(n)^{1/2}\left(\frac{\hbar(km)^{1/2}}{2}\right)^{1/2} = -p_{n-1n} \end{cases}$$

This is the usual phase convention

Phase is a recurrent problem in matrix mechanics because we never look at wavefunctions or evaluate integrals explicitly.

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8. Possible existence of noncommunicating blocks along diagonal of **H, x, p**

you show that $H_{nm} = (n + \mathbf{1/2})\hbar\left(\frac{k}{m}\right)^{1/2}\delta_{nm}$

$$\left(\begin{array}{l}\text{note that } \mathbf{x}^2 \text{ and } \mathbf{p}^2 \text{ have non-zero } \Delta n = \pm 2 \text{ elements but} \\ \frac{1}{2}k\mathbf{x}^2 + \frac{\mathbf{p}^2}{2m} \text{ has cancelling contributions in } \Delta n = \pm 2 \text{ locations}\end{array}\right)$$

This result implies

* all of the possibly independent blocks in **x**, **p**, **H** are identical
* $\varepsilon_i = (1/2)\hbar\omega$ for all i
* degeneracy of all E_n? all same, but can't prove that it is 1.

Creation and Annihilation Operators (CTDL pages 488-508)

* Dimensionless operators
* simple operator algebra rather than complicated real algebra
* matrices arranged according to "selection rules"
* matrix elements calculated by extremely simple rules
* automatic generation of any basis function by repeated operations on lowest (nodeless) basis state

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get rid of system-specific factors of k, μ, ω and also ℏ.

$$\omega = (k/m)^{1/2}$$

$$\underset{\sim}{x} \equiv \left(\frac{m\omega}{\hbar}\right)^{1/2} x \qquad \mathbf{x}^2 = \left(\frac{\hbar}{m\omega}\right)\underset{\sim}{\mathbf{x}}^2$$

dimensionless, regular

$$\underset{\sim}{p} \equiv (\hbar m\omega)^{-1/2} p \qquad \mathbf{p}^2 = \hbar m\omega\, \underset{\sim}{\mathbf{p}}^2$$

We choose these factors to make everything come out dimensionless.

$$\underset{\sim}{H} = \frac{1}{\hbar\omega} H = \frac{1}{2}\left(\underset{\sim}{x}^2 + \underset{\sim}{p}^2\right) \qquad \mathbf{H} = \frac{1}{2}k\mathbf{x}^2 + \frac{\mathbf{p}^2}{2m} = \frac{1}{2}\hbar\omega\left(\underset{\sim}{x}^2 + \underset{\sim}{p}^2\right)$$

$$\left[\underset{\sim}{x}, \underset{\sim}{p}\right] = \left(\frac{m\omega}{\hbar}\frac{1}{\hbar m\omega}\right)^{1/2}[x,p] = \frac{1}{\hbar}(i\hbar) = i$$

from results for **x, p, H**

$$\underset{\sim}{x}_{mn} = 2^{-1/2}\left[(n+1)^{1/2}\delta_{mn+1} + n^{1/2}\delta_{mn-1}\right]$$

$$\underset{\sim}{p}_{mn} = 2^{-1/2} i\left[(n+1)^{1/2}\delta_{mn+1} - n^{1/2}\delta_{mn-1}\right]$$

$$\underset{\sim}{H}_{mn} = (n+1/2)\delta_{mn} \qquad \text{diagonal}$$

square root of larger q.n.

note the negative sign

Kronecker - δ's specify selection rules for nonzero matrix elements

now define something new **a**, $\mathbf{a}^\dagger$ to clean things up even more!

$$\left.\begin{aligned}\mathbf{a} &= 2^{-1/2}\left(\underset{\sim}{\mathbf{x}} + i\,\underset{\sim}{\mathbf{p}}\right)\\ \mathbf{a}^{\dagger} &= 2^{-1/2}\left(\underset{\sim}{\mathbf{x}} - i\,\underset{\sim}{\mathbf{p}}\right)\end{aligned}\right\} \longrightarrow \begin{aligned}\underset{\sim}{\mathbf{x}} &= 2^{-1/2}\left(\mathbf{a} + \mathbf{a}^{\dagger}\right)\\ \underset{\sim}{\mathbf{p}} &= 2^{-1/2}\,i\left(\mathbf{a}^{\dagger} - \mathbf{a}\right)\end{aligned}$$

Let's examine the matrix elements of **a** and $\mathbf{a}^{\dagger}$

$$a_{mn} = \left[2^{-1/2}\,\underset{\sim}{x}_{mn} + 2^{-1/2}\,i\,\underset{\sim}{p}_{mn}\right]$$

$$= \Bigg[\underbrace{\overset{\underset{\sim}{\mathbf{x}}}{\frac{1}{2}(n+1)^{1/2}\delta_{mn+1}} - \overset{i\underset{\sim}{\mathbf{p}}}{\frac{1}{2}(n+1)^{1/2}\delta_{mn+1}}}_{\text{cancel}} + \underbrace{\overset{\underset{\sim}{\mathbf{x}}}{\frac{1}{2}n^{1/2}\delta_{mn-1}} + \overset{i\underset{\sim}{\mathbf{p}}}{\frac{1}{2}n^{1/2}\delta_{mn-1}}}_{\text{add}}\Bigg]$$

group according to "selection rule"

$$\boxed{a_{mn} = n^{1/2}\delta_{mn-1}}$$

first index is <u>one smaller</u> than second

$$a_{mn} = \langle \underset{\text{row}}{m} | \underbrace{\mathbf{a} | \overset{\text{column}}{n}}_{n^{1/2}|n-1\rangle} \rangle$$

a is lowering or "annihilation" operator

similarly

$$\boxed{a^{\dagger}_{mn} = (n+1)^{1/2}\delta_{mn+1}}$$

first index is one larger than second
$\mathbf{a}^{\dagger}$ is a "creation" operator

$$
\mathbf{a}^\dagger = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1^{1/2} & 0 & 0 & 0 & 0 \\
0 & 2^{1/2} & 0 & 0 & 0 \\
0 & 0 & 3^{1/2} & 0 & 0 \\
0 & 0 & 0 & 4^{1/2} & 0
\end{pmatrix}
\end{array}
\qquad
\mathbf{a} = \begin{pmatrix}
0 & 1^{1/2} & 0 & 0 & 0 \\
0 & 0 & 2^{1/2} & 0 & 0 \\
0 & 0 & 0 & 3^{1/2} & 0 \\
0 & 0 & 0 & 0 & 4^{1/2} \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

square root of integers always only one step below main diagonal. **a**, $\mathbf{a}^\dagger$ are obviously not Hermitian

e.g. $\langle 3|\mathbf{a}^\dagger|2\rangle = 3^{1/2}$

$\mathbf{a}^\dagger$ raises

square root of integers always only one step above main diagonal

e.g. $\langle 3|\mathbf{a}|4\rangle = 4^{1/2}$

a lowers

What is so great about **a**, $\mathbf{a}^\dagger$?

$\mathbf{a}|n\rangle = n^{1/2}|n-1\rangle$ annihilates 1 quantum

$\mathbf{a}^\dagger|n\rangle = (n+1)^{1/2}|n+1\rangle$ creates 1 quantum

$|n\rangle = [n!]^{-1/2}\left(\mathbf{a}^\dagger\right)^n|0\rangle$ generate any state from lowest one $|0\rangle$

needed to normalize. Each application of $\mathbf{a}^\dagger$ gives next integer. Do it n times, get n!

more tricks: look at $\mathbf{a}\mathbf{a}^\dagger$ and $\mathbf{a}^\dagger\mathbf{a}$

is $\mathbf{a}\mathbf{a}^\dagger$ Hermitian?

$[(AB)^\dagger = B^\dagger A^\dagger]$ <u>definition of hermitian</u>

$$\left(\mathbf{a}\mathbf{a}^\dagger\right)^\dagger = \mathbf{a}^{\dagger\dagger}\mathbf{a}^\dagger = \mathbf{a}\mathbf{a}^\dagger$$

$\therefore$ $\mathbf{a}\mathbf{a}^\dagger$ and $\mathbf{a}^\dagger\mathbf{a}$ are Hermitian — to what “observable” quantity do they correspond? We will see that one of these is the “number operator.”

$$\mathbf{aa}^\dagger = \frac{1}{2}\left(\underset{\sim}{\mathbf{x}} + \mathbf{i}\,\underset{\sim}{\mathbf{p}}\right)\left(\underset{\sim}{\mathbf{x}} - \mathbf{i}\,\underset{\sim}{\mathbf{p}}\right) = \frac{1}{2}\left(\underset{\sim}{\mathbf{x}}^2 + \mathbf{i}\,\underset{\sim}{\mathbf{p}}\,\underset{\sim}{\mathbf{x}} - \mathbf{i}\,\underset{\sim}{\mathbf{x}}\,\underset{\sim}{\mathbf{p}} + \underset{\sim}{\mathbf{p}}^2\right)$$

$$= \frac{1}{2}\left(\underset{\sim}{\mathbf{x}}^2 + \underset{\sim}{\mathbf{p}}^2 - \mathbf{i}\underbrace{[\underset{\sim}{\mathbf{x}},\underset{\sim}{\mathbf{p}}]}_{\mathbf{i}}\right) = \frac{1}{2}\left(\underset{\sim}{\mathbf{x}}^2 + \underset{\sim}{\mathbf{p}}^2 + 1\right)$$

similarly $\mathbf{a}^\dagger\mathbf{a} = \frac{1}{2}\left(\underset{\sim}{\mathbf{x}}^2 + \underset{\sim}{\mathbf{p}}^2 - 1\right)$

$$\left.\begin{array}{l} \therefore\ \underset{\sim}{\mathrm{H}} = \frac{1}{2}\left(\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}\mathbf{a}^\dagger\right) \quad \text{and} \quad \left[\mathbf{a},\mathbf{a}^\dagger\right] = 1 \\ \underset{\sim}{\mathbf{H}} = \mathbf{a}^\dagger\mathbf{a} + 1/2 \qquad \text{number operator + 1/2} \end{array}\right\} \text{simple form for } \underset{\sim}{\mathrm{H}}$$

$$\mathrm{H} = \hbar\omega\,\underset{\sim}{\mathrm{H}} = \hbar\omega\left(\underset{\uparrow\ \text{\# of quanta}}{\mathbf{a}^\dagger\mathbf{a}} + 1/2\right)$$

$$\mathbf{a}^\dagger\mathbf{a}|\mathrm{n}\rangle = \mathrm{n}|\mathrm{n}\rangle \qquad \mathbf{a}^\dagger\mathbf{a} \text{ is "number operator"}$$

$$\left[\mathbf{aa}^\dagger|\mathrm{n}\rangle = (\mathrm{n}+1)|\mathrm{n}\rangle\right] \qquad \text{not as useful}$$

What have we done? We have exposed all of the “symmetry” and universality of the H–O basis set. We can now trivially work out what the matrix for any $\mathbf{X}^n\mathbf{P}^m$ operator looks like and organize it according to selection rules.

What about $\mathbf{X}^3$?

When you multiply this out, preserve the order of **a** and $\mathbf{a}^\dagger$ factors.

$$\mathbf{X}^3 = \left(\frac{m\omega}{\hbar}\right)^{-3/2} \underset{\sim}{\mathbf{X}}^3$$

$$\underset{\sim}{\mathbf{X}}^3 = \left(2^{-3/2}\right)\left(\mathbf{a}+\mathbf{a}^\dagger\right)^3$$

$$= \left(2^{-3/2}\right)\left[\mathbf{a}^3 + \left(\mathbf{a}^\dagger\mathbf{a}\mathbf{a} + \mathbf{a}\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}\mathbf{a}\mathbf{a}^\dagger\right) + \left(\mathbf{a}\mathbf{a}^\dagger\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{a}\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a}\right) + \mathbf{a}^{\dagger 3}\right]$$

$$\Delta n = \quad -3, \qquad -1, \qquad +1, \qquad +3$$

(# of † minus # of non-†)

Simplify each group using commutation properties so that it has form

$$\mathbf{a}\left[\mathbf{a}^\dagger\mathbf{a}\right]|n\rangle \quad \text{or} \quad \mathbf{a}^\dagger\left[\mathbf{a}^\dagger\mathbf{a}\right]|n\rangle$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$n^{1/2}n|n-1\rangle \qquad (n+1)^{1/2}n|n+1\rangle$$

NONLECTURE: Simplify the $\Delta n = -1$ terms.

$$\mathbf{a}^\dagger\mathbf{a}\mathbf{a} = \mathbf{a}\mathbf{a}^\dagger\mathbf{a}\underbrace{-\mathbf{a}\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}^\dagger\mathbf{a}\mathbf{a}}_{\left[\mathbf{a}^\dagger,\mathbf{a}\right]\mathbf{a}=-\mathbf{a}} = \mathbf{a}\mathbf{a}^\dagger\mathbf{a} - \mathbf{a}$$

add and subtract term needed to reverse order

$$\mathbf{a}\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}\mathbf{a}^\dagger\mathbf{a}\underbrace{-\mathbf{a}\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}\mathbf{a}\mathbf{a}^\dagger}_{\mathbf{a}\left[\mathbf{a},\mathbf{a}^\dagger\right]=\mathbf{a}} = \mathbf{a}\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}$$

$$\left[\mathbf{a}^\dagger\mathbf{a}\mathbf{a} + \mathbf{a}\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}\mathbf{a}\mathbf{a}^\dagger\right] = 3\mathbf{a}\mathbf{a}^\dagger\mathbf{a}$$

try to put everthing into $\mathbf{a}\mathbf{a}^\dagger\mathbf{a}$ order

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NONLECTURE: Simplify the $\Delta n = +1$ terms.

$$\mathbf{a}\mathbf{a}^\dagger\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a}\mathbf{a}^\dagger\underbrace{-\mathbf{a}^\dagger\mathbf{a}\mathbf{a}^\dagger + \mathbf{a}\mathbf{a}^\dagger\mathbf{a}^\dagger}_{\left[\mathbf{a},\mathbf{a}^\dagger\right]\mathbf{a}^\dagger=\mathbf{a}^\dagger} = \mathbf{a}^\dagger\mathbf{a}\mathbf{a}^\dagger + \mathbf{a}^\dagger$$

$$\mathrm{a}^\dagger\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a}\underbrace{-\mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}^\dagger\mathbf{a}\mathbf{a}^\dagger}_{\mathbf{a}^\dagger\left[\mathbf{a},\mathbf{a}^\dagger\right]=\mathbf{a}^\dagger} = \mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}^\dagger$$

$$\left[\mathbf{a}\mathbf{a}^\dagger\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{a}\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a}\right] = 3\mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a} + 3\mathbf{a}^\dagger$$

<u>$\Delta n = \pm 3$</u>

$$\mathbf{a}^3|n\rangle = [n(n-1)(n-2)]^{1/2}|n-3\rangle$$
$$\mathbf{a}^{\dagger 3}|n\rangle = [(n+1)(n+2)(n+3)]^{1/2}|n+3\rangle$$

<u>$\Delta n = \pm 1$</u>

$$\left[\mathbf{a}^\dagger\,\mathbf{aa} + \mathbf{aa}^\dagger\mathbf{a} + \mathbf{aaa}^\dagger\right]|n\rangle = 3\left(n^{3/2}\right)|n-1\rangle$$
$$\left[\mathbf{aa}^\dagger\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{aa}^\dagger + \mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a}\right]|n\rangle = 3\left[(n+1)^{1/2}n|n+1\rangle + (n+1)^{1/2}|n+1\rangle\right]$$
$$= 3(n+1)^{1/2}(n+1)|n+1\rangle$$
$$= 3(n+1)^{3/2}|n+1\rangle$$

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no need to do matrix multiplication. Just play with $\mathbf{a}$, $\mathbf{a}^\dagger$ and commutation rule and $\mathbf{a}^\dagger\mathbf{a}$ number operator

"Second Quantization"

same as $|n+3\rangle\langle n|$

$$\begin{aligned}\mathbf{x}^3_{mn} = \left(\frac{\hbar}{2m\omega}\right)^{3/2}\Big[&\delta_{mn+3}\left((n+1)(n+2)(n+3)\right)^{1/2}\\ &+ \delta_{mn+1}\; 3(n+1)^{3/2} &&\leftarrow |n+1\rangle\langle n|\\ &+ \delta_{mn-1}\; 3n^{3/2} &&\leftarrow |n\rangle\langle n-1|\\ &+ \delta_{mn-3}\; \left(n(n-1)(n-2)\right)^{1/2}\Big]\end{aligned}$$

$|n\rangle\langle n-3|$

simple! $\mathbf{x}^3$ is arranged into four separate terms, each with its own explicit selection rule.

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* $V(x) = \frac{1}{2}k\mathbf{x}^2 + \underbrace{a\mathbf{x}^3 + b\mathbf{x}^4}_{\text{anharmonic terms} \rightarrow \text{perturbation theory}}$

* IR transition intensities $\propto \left|\langle n|\mathbf{x}|n+1\rangle\right|^2$

* Survival and transfer probabilities of initially prepared pure harmonic oscillator non-eigenstate in anharmonic potential.

* Expectation values of any function of x and p.

Universitality: all k,m (system-specific) constants are removed until we put them back in at the end of the calculation.

e.g., What is $\langle \Delta x \rangle^2 = \left[\left\langle x^2 \right\rangle - \langle x \rangle^2 \right]$

$$\mathbf{x}^2 = \frac{\hbar}{m\omega} \underset{\sim}{x} = \frac{\hbar}{m\omega} \left[\frac{1}{2} \left(\mathbf{a} + \mathbf{a}^\dagger \right)^2 \right]$$

$$\langle \Delta x \rangle^2 = \frac{\hbar}{2m\omega} \left[\left\langle \left(\mathbf{a} + \mathbf{a}^\dagger \right)^2 \right\rangle - \left\langle \mathbf{a} + \mathbf{a}^\dagger \right\rangle^2 \right] ?$$

pure numbers in [ ]