5.72 Problem Set #5 (Spring 2012) Cao

I. Density fluctuations

1) Derive the equilibrium value of density fluctuations
$$S(k) = \frac{1}{N}\langle|\delta\rho_k|^2\rangle = 1 + \rho_0\hat{h}(k)$$
2) $S(k=0) = \rho_0 k_B T \chi_T$, where
$$\chi_T = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial P}\right)_T$$
is the isothermal compressibility.
3) Show 2 Re$\left[\hat{F}(k, z = i\omega)\right] = S(k,\omega)$ where $\hat{F}(k,z) = \int_0^\infty e^{-zt}F(k,t)dt$ is the Laplace transform.
4) To first order in k, we obtain
$$F(k,t) = S(k)\left[\left(1-\frac{1}{\gamma}\right)e^{-ak^2t/\gamma} + \frac{1}{\gamma}\cos(kc_s t)e^{-k^2\Gamma t}\right].$$
Use the expression for $F(k,t)$ to derive $S(k,\omega)$.
5) Make a plot of $S(k,\omega)$ for a typical value of k. (Ref: Reichl p 544-562)

II. The transverse current is defined as
$$J_k = \sum_j^N v_{jx}(t)e^{-ikz_j(t)}.$$
1) Show $\langle|J_k|^2\rangle = \frac{Nk_BT}{M}$ (note $\langle v_{ix}v_{jx}\rangle = \delta_{ij}\langle v_x^2\rangle$)
2) Use the Navier-Stokes equation to show $J_k(t) = J_k(0)e^{-\nu k^2 t}$ where $\nu = \frac{\eta}{\rho_0 m}$.
3) Prove
$$\sum_{ij}\langle v_{ix}(t)v_{jx}(0)\left[z_i(t) - z_j(0)\right]^2\rangle = \langle|A(t)-A(0)|^2\rangle\frac{1}{m^2}$$
when $A(t) = m\sum_i v_{ix}(t)z_i(t)$.
4) *Derive
$$\eta = \frac{1}{V\,k_BT}\lim_{t\to\infty}\frac{1}{2t}\langle|A(t)-A(0)|^2\rangle.$$
5) Derive
$$\eta = \frac{1}{V\,k_BT}\int_0^\infty\langle\sigma_{xz}(t)\sigma_{xz}(0)\rangle dt\,,$$
where $\sigma_{xz} = \frac{d}{dt}A$.
6) *Show $\langle\sigma_{\alpha\beta}\rangle = PV\delta_{\alpha\beta}$ (Ref. 5.70 notes, McQuarrie p513-525)

III. Using the mean-free path approximation [McQuarrie p. 523], derive expressions for D and η in terms of the average collision time t_c.

IV. Landau-Placzek ratio (ref. Reichl) f_{L-P}

$$f_{L-P} = \frac{\langle \Delta\rho^2 \rangle_{\text{thermal}}}{\langle \Delta\rho^2 \rangle_{\text{mechanical}}} = \frac{\left(\frac{\partial \rho}{\partial S}\right)_P^2 \langle \Delta S^2 \rangle}{\left(\frac{\partial \rho}{\partial P}\right)_S^2 \langle \Delta P^2 \rangle}.$$

1) Explain this expression.
2) Using the thermal fluctuation relation $P \propto \exp\left\{-\frac{\beta}{2}[\Delta S \Delta T - \Delta P \Delta V]\right\}$ show

$$\frac{\langle \Delta S^2 \rangle}{\langle \Delta P^2 \rangle} = -\frac{\left(\frac{\partial S}{\partial T}\right)_P}{\left(\frac{\partial P}{\partial V}\right)_S}.$$

3) Derive

$$f_{L-P} = \left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial P}{\partial S}\right)_\rho.$$

4) *Show $f_{L-P} = \gamma - 1 = \frac{C_P - C_V}{C_V}$ when $\gamma = \frac{C_P}{C_V}$.

V. *Derivations
1) To second order in k, derive

$$F(k,t) = \left(1 - \frac{1}{\gamma}\right) e^{-ak^2 t/\gamma} + \frac{1}{\gamma} e^{-\Gamma k^2 t}[\cos(c_s kt) + d(k)\sin(c_s kt)],$$

where $d(k) = \frac{\Gamma + \left(1 - \frac{1}{\gamma}\right)a}{c_s} k$.

2) Prove

$$b = \frac{m\beta}{2} \lim_{\omega \to 0} \lim_{k \to 0} \left(\frac{\omega}{k}\right)^4 S(k,\omega)$$

when $b = \frac{1}{m\rho_0}\left(\eta_B + \frac{4}{3}\eta\right)$.

VI. *Derive the explicit expression for the intensity of scattered light from classical electromagnetic theory. (Ref. Reichl)

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