Lecture Notes 1: Analyzing the Shower Process

what we will be doing in this module

The broad context for this module is the commonsense notion that, when designing something, one should plan for the off-normal conditions that may occur.

The particular context is the design of continuous chemical processes. Design starts with some product in mind (a simple molecule, a complex substance, or a structured product), and possibly several distinctive paths to reach it (the variations in chemistry, sequence, and processing that comprise a chemical process). The alternative processes are examined in a cursory fashion and unpromising ones discarded. As the design proceeds, increasing effort is spent on fewer alternatives in the approach to detailed design.

A continuous process is conceived and designed as a steady-state operation. However, the process must start up, shut down, and operate in the event of disturbances, and so the time-varying behavior of the process should not be neglected. A proper dynamic simulation of the process requires that a number of design details be in place, and thus must take place in later stages of design. Even so, it is helpful to consider the operability of a process earlier in the design, when alternatives are still being compared. In this module, we will examine some tools that will help to evaluate the operability of the candidate process at the preliminary design stage, before substantial effort has been invested. Thus, these are screening tools.

The ideas presented in these notes are derived from the texts by Marlin (1), McAvoy (2), and Seider et al (3).

a few ways in which processes can go wrong

bad operation

- inadequate procedures
- mis-tuned controllers
- malfunctioning instruments

bad implementation of the design

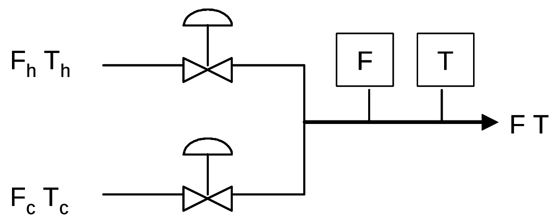
- backwards flowmeters and other installation mistakes
- valve in wrong place
- wrong type of valve
- mistakes in construction/check-off
- field decisions that should have been specified in the design

bad design

- mis-sized equipment
- poor controller scheme
- prone to instability
- poorly located or selected instrument
- insufficient number of instruments

we will begin by analyzing a simple process

It’s only an ordinary shower, but we’ll dress it up as a chemical process: two feed streams enter and mix to form an outlet stream of flow F and temperature T. Instruments are provided on the exit line to measure these quantities, and valves regulate the feed streams.



Our shower should operate as a steady process. To design it, we must specify steady conditions, valve sizes, pipe size, supply pressure, instruments, and so forth. Mindful of the discussion above, we'll also want to examine its operability.

we think about how the process should operate
The process operating objective is to maintain steady F and T at desired conditions. What might interfere with that objective? How might we respond?

It will clarify our thinking if we classify the variables:

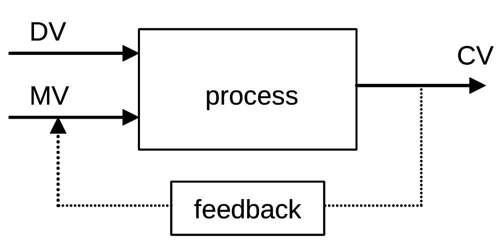
CV – controlled variables F T (variables important for safety, product quality, etc.)
MV – manipulated variables F_h F_c (manipulate to exert influence on CV)
DV – disturbance variables T_h T_c (these disrupt CV, and we try to counter them with MV)

When we consider using MV to counter DV so that CV can be maintained at a set point, we introduce the topic of process control.

we briefly describe the notion of process control
When we talk about process control, we usually run across a feedback structure:

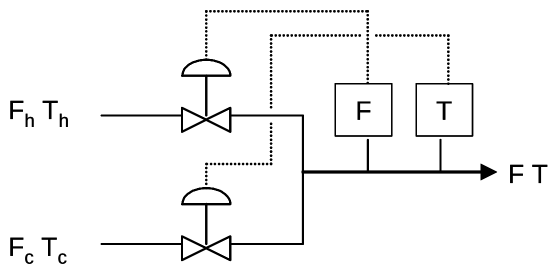
observe CV and use that information to adjust MV to compensate for the effects of DV.

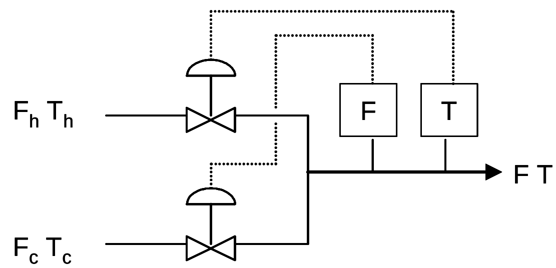
"Feedback" implies a reverse flow: while inputs DV and MV flow through the process to affect the output CV, information about CV is fed back, outside the process, to change MV. The feedback path creates a "control loop".



Deciding how to adjust MV at any time is the function of a device called the "controller". In "automatic control" we "close the loop", giving the controller authority to set MV. If we "open the loop" we block the feedback path and thus set MV manually

we apply feedback control to the shower process
Here are two alternative feedback structures for the shower:





Each structure comprises two separate control loops; i.e., connections between a controlled and a manipulated variable. In the left figure, for example, if the flow rate is observed to change, the hot water valve will be adjusted. Changes in the temperature motivate adjustments to the cold water valve. The right figure makes the opposite pairings between CV and MV.

we must choose one of these alternatives to pursue
These are design alternatives, and we must choose between them. Steady-state considerations seem not to favor one over the other, so we must ask which of these is better in overcoming a disturbance? responding to a change in set point? maintaining a stable operation? reducing interaction between the two control loops?

We have several days of mathematics to do in answering this question. Before we start that, though, we should use our intuition: which do you prefer? Why?

chemical engineers frequently find recourse in material and energy balances
material balance:

$$\frac{d}{dt}\left[\int_V \rho dV\right] = F_h\rho_h + F_c\rho_c - F\rho \qquad (1\text{-}1)$$

energy balance:

$$\frac{d}{dt}\left[\int_V \rho C_p\left(T - T_{ref}\right)dV\right] = F_h\rho_h C_{ph}\left(T_h - T_{ref}\right) + F_c\rho_c C_{pc}\left(T_c - T_{ref}\right) - F\rho C_p\left(T - T_{ref}\right) \qquad (1\text{-}2)$$

We'll make two approximations to simplify the analysis:

(1) physical properties constant with temperature; e.g. $\rho = \rho_c = \rho_h$ and thus

$$\frac{d}{dt}\left[\int_V \rho dV\right] = \frac{d}{dt}\left[\rho V\right] = 0 \qquad (1\text{-}3)$$

This is often a reasonable approximation for liquids. It implies that the amount of mass in the system is constant, and that changes in flowrate are immediately communicated throughout the system.

(2) V is not significantly large, so that the enthalpy variations within V as T varies are not significant. Thus

$$\frac{d}{dt}\left[\int_V \rho C_p\left(T - T_{ref}\right)dV\right] \cong 0 \qquad (1\text{-}4)$$

This is a more serious assumption. One of its implications is that there is no time delay: a change in valve position or inlet temperature immediately affects the flow and temperature at the showerhead.

With these simplifications (1-1) and (1-2) become

$$\begin{aligned} F &= F_h + F_c \\ T &= \frac{F_h}{F_h + F_c} T_h + \frac{F_c}{F_h + F_c} T_c \end{aligned} \qquad (1\text{-}5)$$

These equations (under our assumptions) apply whether the flow is steady or time-varying.

we make a linear approximation to the nonlinear energy balance
We linearize the energy balance because

- it makes the problem easier to solve
- our control objective is to operate at a particular point. A linear approximation based on that operating point can be quite accurate in the region of that point. (Of course, far from the reference point, it can be wildly misleading!)

We express a nonlinear function f as a Taylor series anchored on the reference conditions x_r and y_r; we truncate after the first derivatives.

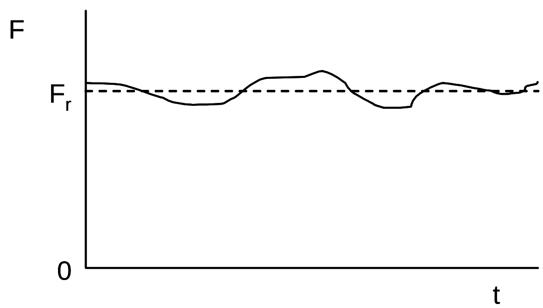
$$f\left(x_1, x_2\right) \approx f\left(x_{1r}, x_{2r}\right) + \left.\frac{\partial f}{\partial x_1}\right|_r \left(x_1 - x_{1r}\right) + \left.\frac{\partial f}{\partial x_2}\right|_r \left(x_2 - x_{2r}\right) \qquad (1\text{-}6)$$

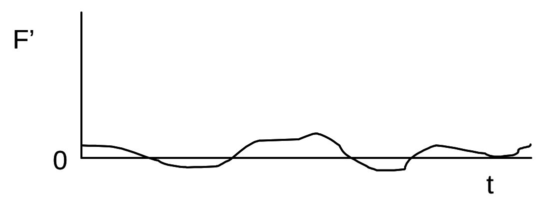
This Taylor series can be extended to other numbers of variables. In our case, the shower temperature is a function of the variables F_h, F_c, T_h, and T_c. Four partial derivatives will be necessary to approximate this function. Doing so, we arrive at a linearized energy balance

$$T = \frac{F_{hr}T_{hr} + F_{cr}T_{cr}}{F_{hr} + F_{cr}} + \left[\frac{T_{hr}}{F_{hr} + F_{cr}} - \frac{F_{hr}T_{hr} + F_{cr}T_{cr}}{(F_{hr} + F_{cr})^2}\right](F_h - F_{hr}) + \frac{F_{hr}}{F_{hr} + F_{cr}}(T_h - T_{hr})$$
$$+ \left[\frac{T_{cr}}{F_{hr} + F_{cr}} - \frac{F_{hr}T_{hr} + F_{cr}T_{cr}}{(F_{hr} + F_{cr})^2}\right](F_c - F_{cr}) + \frac{F_{cr}}{F_{hr} + F_{cr}}(T_c - T_{cr}) \qquad (1\text{-}7)$$

we introduce deviation variables to focus on departure from desired conditions
Deviation variables simplify the equations, as well as focus on the disturbances.

$$F \equiv F_r + F' \qquad F_h \equiv F_{hr} + F'_h \qquad F_c \equiv F_{cr} + F'_c$$
$$T \equiv T_r + T' \qquad T_h \equiv T_{hr} + T'_h \qquad T_c \equiv T_{cr} + T'_c \qquad (1\text{-}8)$$





The reference condition is chosen to satisfy the steady state M&EB and to be the desired, or set point, condition. Therefore, only four of the six reference values can be independently chosen.

$$F_r = F_{hr} + F_{cr}$$
$$T_r = \frac{F_{hr}T_{hr} + F_{cr}T_{cr}}{F_{hr} + F_{cr}} \qquad (1\text{-}9)$$

We now substitute the deviation variables into the material balance (1-5), and simplify.

$$F = F_h + F_c$$
$$F_r + F' = F_{hr} + F'_h + F_{cr} + F'_c \qquad (1\text{-}10)$$
$$F' = F'_h + F'_c$$

Similarly for the energy balance (1-5),

$$T' = \left[\frac{T_{hr} - T_r}{F_r}\right]F'_h + \left[\frac{T_{cr} - T_r}{F_r}\right]F'_c + \frac{F_{hr}}{F_r}T'_h + \frac{F_{cr}}{F_r}T'_c \qquad (1\text{-}11)$$

we introduce scaled variables to put deviations into perspective

The fuel gauge in a car tells you the fraction of the fuel remaining, not the actual volume. One litre may be sufficient to get you home in your Prius, but not in your Escalade – the gauge will indicate the relevance of 1L to each automobile. This is the use of a scaled variable.

Suppose we expect T_h to vary within the range $T_{h,min}$ to $T_{h,max}$. Then we define a scaled variable

$$T_h^{*'} \equiv \frac{T_h - T_{hr}}{\Delta T_h} \qquad \Delta T_h = T_{h,max} - T_{h,min} \tag{1-12}$$

$$\text{at minimum}: T_h^{*'}\Big|_{min} = \frac{T_{h,min} - T_{hr}}{\Delta T_h} \qquad -1 \le T_h^{*'}\Big|_{min} \le 0$$
$$\text{at maximum}: T_h^{*'}\Big|_{max} = \frac{T_{h,max} - T_{hr}}{\Delta T_h} \qquad 0 \le T_h^{*'}\Big|_{max} \le 1 \tag{1-13}$$

Where the maximum and minimum scaled values actually lie depends on where the reference value T_{hr} falls within the operating range ΔT_h.

- for academic derivations and textbook material, we tend to set the reference to the desired value (the set point) as in the derivation above. Thus the scaled variable would vary around zero (from -0.5 to 0.5, say).
- for operation in the control room, we prefer a 0 to 1 scale (or equivalently, 0 to 100%) so we set the reference to the minimum value. Then the desired value will become, e.g., 0.5 in scaled terms, and we will see the scaled variable wander about that value.

Now, substitute scaled variable definitions into the material and energy balances (1-10) and (1-11).

$$F^{*'} = \frac{\Delta F_h}{\Delta F} F_h^{*'} + \frac{\Delta F_c}{\Delta F} F_c^{*'} \tag{1-14}$$

$$T^{*'} = \left[\frac{T_{hr} - T_r}{F_r}\right] \frac{\Delta F_h}{\Delta T} F_h^{*'} + \left[\frac{T_{cr} - T_r}{F_r}\right] \frac{\Delta F_c}{\Delta T} F_c^{*'} + \frac{F_{hr}}{F_r}\frac{\Delta T_h}{\Delta T} T_h^{*'} + \frac{F_{cr}}{F_r}\frac{\Delta T_c}{\Delta T} T_c^{*'} \tag{1-15}$$

don't panic when you look at these equations

They may look complicated, but they are the same old material and energy balances: the flow rate is the still the sum of the inlet flows, and the outlet temperature depends on the inlet flows and temperatures. The * means that the physical variable has been divided by its operating range, so that its magnitude will not exceed 1. The ′ means that a steady condition has been subtracted from the physical variable as a reference. The coefficients of these scaled deviation variables are constants, made up of the partial derivatives from linearizing and ratios of the scaling ranges.

how to choose scaling ranges for the variables

- disturbances are imposed on you. Scale according to what you anticipate (from operating data, similar operations, general experience, or judgment.)
- controlled variables are what you hope to achieve. Scale according to some realistic range of operation.
- manipulated variables are a design choice. Scale by the amount of influence you must bring to bear to counteract the disturbances.

One of the objectives of our modeling is to reconcile these various specifications.

linear equation systems are often presented in matrix notation

$$\begin{bmatrix} F^{*'} \\ T^{*'} \end{bmatrix} = \begin{bmatrix} \dfrac{\Delta F_h}{\Delta F} & \dfrac{\Delta F_c}{\Delta F} \\ \dfrac{T_{hr} - T_r}{F_r}\dfrac{\Delta F_h}{\Delta T} & \dfrac{T_{cr} - T_r}{F_r}\dfrac{\Delta F_c}{\Delta T} \end{bmatrix} \begin{bmatrix} F_h^{*'} \\ F_c^{*'} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \dfrac{F_{hr}}{F_r}\dfrac{\Delta T_h}{\Delta T} & \dfrac{F_{cr}}{F_r}\dfrac{\Delta T_c}{\Delta T} \end{bmatrix} \begin{bmatrix} T_h^{*'} \\ T_c^{*'} \end{bmatrix} \qquad (1\text{-}16)$$

We shall give these matrices standard names, because these linear system models all look alike:

$$\underline{y}^{*'} = \underline{\underline{P}}_m^{*}\,\underline{x}_m^{*'} + \underline{\underline{P}}_d^{*}\,\underline{x}_d^{*'} \qquad (1\text{-}17)$$

When we make a linear approximation to some process model, it always comes down to this form. The gain matrices $\underline{\underline{P}}_m^*$ and $\underline{\underline{P}}_d^*$ contain the numbers that distinguish one process from another. The input vectors $\underline{x}_m^{*'}$ and $\underline{x}_d^{*'}$ are processed by the gain matrices to produce the output vector $\underline{y}^{*'}$.

We could of course write the linear system equation without the * superscript if we had not scaled the variables. However, the ′ superscript is necessary, because the linear system approximation depends on deviations around a reference condition.

we introduce the Relative Gain Array to help us make a control decision

We now have a very distinguished-looking material and energy balance pair. We must use our linear system model to answer the question of how to pair up the controlled and manipulated variables into control loops. The RGA (Relative Gain Array) can help. The RGA builds on the gain matrix $\underline{\underline{P}}_m^*$ to illustrate how manipulated variables influence controlled variables when we connect them by control loops. Thus, it indicates how control loops might interact – how attempting to control one variable might disturb another as a side-effect.

$$\underline{\underline{\Lambda}} \equiv \begin{array}{c} \\ (y_1) \\ (y_2) \\ (y_3) \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} (x_{m1}) & (x_{m2}) & (x_{m3}) & \cdots \end{array} \\ \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \cdots \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \\ \cdots & & & \cdots \end{bmatrix} \end{array} \tag{1-18}$$

(The variable names in parentheses are labels. They show that, e.g., coefficient λ_{31} relates manipulated variable x_{m1} to controlled variable y_3.)

The RGA elements are defined as a ratio of gains.

$$\lambda_{ij} \equiv \frac{\text{gain of the } x_{mj} \rightarrow y_i \text{ loop with other loops open}}{\text{gain of the } x_{mj} \rightarrow y_i \text{ loop with other loops perfect}} = \frac{\left.\dfrac{\partial y_i}{\partial x_{mj}}\right|_{x_{mk} \neq x_{mj} \text{ constant}}}{\left.\dfrac{\partial y_i}{\partial x_{mj}}\right|_{y_k \neq y_i \text{ constant}}} \tag{1-19}$$

For the numerator, all control loops except $x_{mj} - y_i$ have been disconnected (made open). Therefore there is no mechanism for manipulations of x_{mj} to feed back through other control loops and motivate changes in the other x_{mk}. For the denominator, it is imagined that all the other loops have been connected (made closed), and furthermore that they are SO GOOD that the other y_k never vary away from set point, even when x_{mj} is manipulated to influence y_i.

what these RGA elements mean

- if $\lambda_{ij} = 1$, then x_{mj} influences y_i with no interference from the other control loops. This is really good.
- if $\lambda_{ij} = 0$, then x_{mj} has little effect on y_i. It will not be of much use in controlling it. In fact, its main effects will be exerted through the other loops, so there will be significant loop interaction.
- if λ_{ij} = big, then the effect of x_{mj} is greatly diluted by the other loops. Changes in those loops will cause the influence of x_{mj} on y_i to vary widely, causing stability problems.
- if $\lambda_{ij} < 0$, the operation of other loops reverses the effect of x_{mj}. Stability problems!

As λ_{ij} departs from 1, the behavior of the loop is subject to non-welcome influences.

we use RGA to decide how to pair CV with MV

The RGA summarizes all possible MV-CV pairs. Our design decision is to select n pairs from the n×n matrix of choices – we might think of circling our picks on a printed page. In the end, each row or column will have only one circled element.

a rule for pairing CV and MV, based on the RGA
McAvoy (2, p.84) says

> "Always pair on positive RGA elements that are closest to 1.0. Check the resulting pairings for stability using Niederlinski's theorem. If the pairings are unstable choose other positive pairings with values closest to 1.0. Avoid negative pairings if possible."

He illustrates his instability caveat with a case in which RGA elements of 1.0 are discarded in favor of elements equal to 4.5. The Niederlinski Stability Theorem (modified from (2), p.83) is

The closed loop system resulting from the pairing

$$x_{m1} - y_1 \quad x_{m2} - y_2 \quad \ldots \quad x_{mn} - y_n$$

is unstable if

$$\frac{\det\left(\underline{\underline{P}}_m\right)}{\prod_{i=1}^{n} p_{ii}} < 0 \qquad (1\text{-}20)$$

where p_{ii} are the diagonal elements of $\underline{\underline{P}}_m$. Notice that the gain matrix $\underline{\underline{P}}_m$ must be arranged so that the RGA pairing is on the diagonal.

we can calculate the individual RGA elements from the definition
The linear system model equations can be solved for individual RGA elements by holding various x_m and y variables constant, according to the definition (1-19).

we can also calculate the RGA directly from the gain coefficient matrix

$$\underline{\underline{\Lambda}} = \underline{\underline{P}}_m \otimes \left(\underline{\underline{P}}_m^{\ -1}\right)^T \qquad (1\text{-}21)$$

Let's try it for the general 2×2 case: first we do the matrix operations on the gain coefficient matrix.

$$
\begin{aligned}
&\underline{\underline{P}}_m = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \\
&\det\left(\underline{\underline{P}}_m\right) = p_{11}p_{22} - p_{12}p_{21} \\
&\mathrm{adj}\left(\underline{\underline{P}}_m\right) = \begin{bmatrix} p_{22} & -p_{21} \\ -p_{12} & p_{11} \end{bmatrix} \\
&\mathrm{inv}\left(\underline{\underline{P}}_m\right) = \underline{\underline{P}}_m^{-1} = \frac{\left(\mathrm{adj}\left(\underline{\underline{P}}_m\right)\right)^T}{\det\left(\underline{\underline{P}}_m\right)} = \frac{\begin{bmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{bmatrix}}{p_{11}p_{22} - p_{12}p_{21}}
\end{aligned}
\tag{1-22}
$$

Then we perform the element-by-element product.

$$
\begin{aligned}
\underline{\underline{\Lambda}} &= \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \otimes \begin{bmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{bmatrix} \frac{1}{p_{11}p_{22} - p_{12}p_{21}} \\
&= \begin{bmatrix} p_{11}p_{22} & -p_{12}p_{21} \\ -p_{12}p_{21} & p_{11}p_{22} \end{bmatrix} \frac{1}{p_{11}p_{22} - p_{12}p_{21}}
\end{aligned}
\tag{1-23}
$$

This is the form of the RGA for a 2×2 system. What it tells depends on the numerical values for p_{ij}, which (of course) depend on the particular process being modeled.

our shower process is a 2×2 system, so substitute the elements from $\underline{\underline{P}}_m^*$ into the form

$$
\begin{aligned}
\underline{\underline{\Lambda}} &= \begin{bmatrix} \frac{\Delta F_h}{\Delta F}\frac{\Delta F_c}{\Delta T}\frac{T_{cr}-T_r}{F_r} & -\frac{\Delta F_h}{\Delta T}\frac{\Delta F_c}{\Delta F}\frac{T_{hr}-T_r}{F_r} \\ -\frac{\Delta F_h}{\Delta T}\frac{\Delta F_c}{\Delta F}\frac{T_{hr}-T_r}{F_r} & \frac{\Delta F_h}{\Delta F}\frac{\Delta F_c}{\Delta T}\frac{T_{cr}-T_r}{F_r} \end{bmatrix} \frac{1}{\frac{\Delta F_h}{\Delta F}\frac{\Delta F_c}{\Delta T}\frac{T_{cr}-T_r}{F_r} - \frac{\Delta F_h}{\Delta T}\frac{\Delta F_c}{\Delta F}\frac{T_{hr}-T_r}{F_r}} \\
&= \begin{bmatrix} \frac{T_r - T_{cr}}{T_{hr} - T_{cr}} & \frac{T_{hr} - T_r}{T_{hr} - T_{cr}} \\ \frac{T_{hr} - T_r}{T_{hr} - T_{cr}} & \frac{T_r - T_{cr}}{T_{hr} - T_{cr}} \end{bmatrix}
\end{aligned}
\tag{1-24}
$$

We find

- Each element has a value between 0 and 1, depending on how T_r falls between T_{cr} and T_{hr}.
- All of the scaling factors divided out – the RGA is inherently dimensionless (recall its definition as a ratio). This is why the * scaling marker was omitted from the foregoing matrix manipulations of the 2×2 RGA.

We can use the M&EB at the reference condition (1-9)

$$F_r = F_{hr} + F_{cr}$$
$$T_r = \frac{F_{hr} T_{hr} + F_{cr} T_{cr}}{F_{hr} + F_{cr}}$$

to express the RGA elements (1-24) in terms of the manipulated variables

$$\frac{T_r - T_{cr}}{T_{hr} - T_{cr}} = \frac{F_{hr}}{F_r} = \gamma$$
$$\frac{T_{hr} - T_r}{T_{hr} - T_{cr}} = \frac{F_{cr}}{F_r} = 1 - \gamma \qquad (1\text{-}25)$$

where we have defined γ as the fraction of the flow that is from the hot water supply. The RGA then becomes

$$\underline{\underline{\Lambda}} = \begin{matrix} & (F_h) & (F_c) \\ (F) & \gamma & 1-\gamma \\ (T) & 1-\gamma & \gamma \end{matrix} \qquad (1\text{-}26)$$

remember that our question is which way to pair CV with MV
We have two choices:

(1) manipulate hot water flow to control total flow, and adjust temperature by manipulating cold water.

$$\underline{\underline{\Lambda}} = \begin{matrix} & (F_h) & (F_c) \\ (F) & \boxed{\gamma} & 1-\gamma \\ (T) & 1-\gamma & \boxed{\gamma} \end{matrix} \qquad (1\text{-}27)$$

(2) manipulate cold water to control total flow, and adjust temperature by manipulating hot water.

$$\underline{\underline{\Lambda}} = \begin{matrix} & (F_h) & (F_c) \\ (F) & \gamma & \boxed{1-\gamma} \\ (T) & \boxed{1-\gamma} & \gamma \end{matrix} \qquad (1\text{-}28)$$

We want pairings that have RGA elements close to 1. Therefore, if we use mostly hot water in the mix (F_r is mostly F_{hr}), then $\gamma \sim 1$ and we should choose pairing (1). Should conditions be opposite, we should use pairing (2). As γ approaches 0.5, neither pairing would be preferred, and either would result in significant interaction between the control loops.

we introduce the Disturbance Cost to see if our manipulated variables are strong enough
After we have used the RGA to guide our deployment of MV, we must confirm that they are strong enough to do the job. Recall the approximate linear model of our system:

$$\underline{y}^{*'} = \underline{\underline{P}}_m^* \underline{x}_m^{*'} + \underline{\underline{P}}_d^* \underline{x}_d^{*'} \qquad (1\text{-}29)$$

If control is perfect, all CV will remain at set point, regardless of how the disturbances $\underline{x}_d^{*'}$ vary. then $\underline{y}' = 0$. Substituting into (1-29)

$$0 = \underline{\underline{P}}_m^* \underline{x}_{m,pc}^{*'} + \underline{\underline{P}}_d^* \underline{x}_d^{*'} \qquad (1\text{-}30)$$

Achieving perfect control requires that the manipulated variables adjust to values $\underline{x}_{m,pc}^{*'}$ that compensate for $\underline{x}_d^{*'}$. This manipulative action must satisfy the governing material and energy balances, so that

$$\underline{x}_{m,pc}^{*'} = -\underline{\underline{P}}_m^{*\,-1} \underline{\underline{P}}_d^* \underline{x}_d^{*'} \qquad (1\text{-}31)$$

If we view the change of a manipulated variable as the 'cost' of responding to a disturbance, we can summarize the overall 'cost' of perfect control by calculating the norm, or magnitude, of the $\underline{x}_{m,pc}^{*'}$ vector. Lewin (in (3)) calls it the Disturbance Cost (DC).

$$DC = \text{norm}\left(\underline{x}_{m,pc}^{*'}\right) \qquad (1\text{-}32)$$

the standard caution about believing too much in fancy-looking math
We recognize that the linear model is accurate only near the reference point, so that computations in which disturbance and manipulated variables are pushed to their maximum deviations are unreliable as quantitative measures. However, for screening candidates during preliminary design, the DC offers an indication of controllability that may be sufficient to identify problems for further examination. That is, if we do not like $\underline{x}_{m,pc}^{*'}$, we must reexamine our assumptions about $\underline{x}_d^{*'}$, reevaluate the scaling ranges we used for $\underline{x}_m^{*'}$, or alter the process ($\underline{\underline{P}}_m^*$ and $\underline{\underline{P}}_d^*$).

we derive DC for our shower example, obtaining an analytic expression
To express DC for the shower, we must invert $\underline{\underline{P}}_m^*$ and multiply it by $\underline{\underline{P}}_d^*$.

$$\underline{\underline{P_m^*}}^{-1}\underline{\underline{P_d^*}} = \begin{bmatrix} \dfrac{T_r - T_{cr}}{T_{hr} - T_{cr}}\dfrac{\Delta F}{\Delta F_h} & \dfrac{F_r}{T_{hr} - T_{cr}}\dfrac{\Delta T}{\Delta F_h} \\ \dfrac{T_{hr} - T_r}{T_{hr} - T_{cr}}\dfrac{\Delta F}{\Delta F_c} & \dfrac{-F_r}{T_{hr} - T_{cr}}\dfrac{\Delta T}{\Delta F_c} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \dfrac{F_{hr}}{F_r}\dfrac{\Delta T_h}{\Delta T} & \dfrac{F_{cr}}{F_r}\dfrac{\Delta T_c}{\Delta T} \end{bmatrix}$$
$$= \begin{bmatrix} \dfrac{F_{hr}}{T_{hr} - T_{cr}}\dfrac{\Delta T_h}{\Delta F_h} & \dfrac{F_{cr}}{T_{hr} - T_{cr}}\dfrac{\Delta T_c}{\Delta F_h} \\ \dfrac{-F_{hr}}{T_{hr} - T_{cr}}\dfrac{\Delta T_h}{\Delta F_c} & \dfrac{-F_{cr}}{T_{hr} - T_{cr}}\dfrac{\Delta T_c}{\Delta F_c} \end{bmatrix} \quad (1\text{-}33)$$

Then

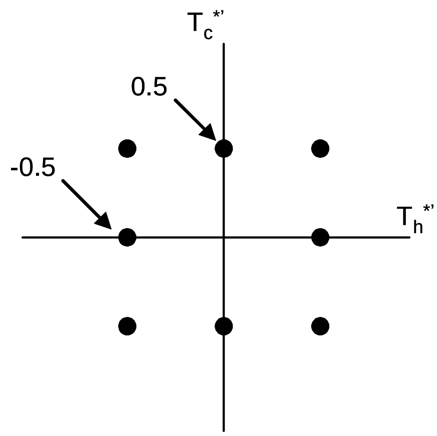
$$\begin{bmatrix} F_{h,pc}^{*'} \\ F_{c,pc}^{*'} \end{bmatrix} = \underline{\underline{P_m^*}}^{-1}\underline{\underline{P_d^*}}\begin{bmatrix} T_h^{*'} \\ T_c^{*'} \end{bmatrix}$$
$$= \frac{F_{hr}\Delta T_h T_h^{*'} + F_{cr}\Delta T_c T_c^{*'}}{T_{hr} - T_{cr}}\begin{bmatrix} \dfrac{-1}{\Delta F_h} \\ \dfrac{1}{\Delta F_c} \end{bmatrix} \quad (1\text{-}34)$$

To obtain DC, we take the norm of this vector.

$$DC = \text{norm}\left(\begin{bmatrix} F_{h,pc}^{*'} \\ F_{c,pc}^{*'} \end{bmatrix}\right) = \left|\frac{F_{hr}\Delta T_h T_h^{*'} + F_{cr}\Delta T_c T_c^{*'}}{T_{hr} - T_{cr}}\right|\left(\frac{1}{\Delta F_h^{\ 2}} + \frac{1}{\Delta F_c^{\ 2}}\right)^{1/2} \quad (1\text{-}35)$$

DC tells us which disturbance conditions are the worst
DC is a function of the disturbance input. We can see from (1-35) that the largest DC occurs when both cold and hot inlet temperatures change in the same direction. In such a case, the temperature controller must counter a change in temperature by using a stream whose temperature has changed in the direction of the disturbance. By contrast, opposite-direction temperature changes are self-compensating and therefore not as costly to mitigate.

It is useful to compute DC over the domain of possible disturbances. For illustration, we can assume that the reference value is in the middle of the operating range for each disturbance variable, such that each scaled deviation variable varies between -0.5 and 0.5. Presuming that the disturbance variables may change independently, we should examine the extremes in all combinations. (In an actual design, the reference values and operating ranges may not lead to this symmetric domain.)



For plotting the effects of two disturbance variables, we can summarize the two variables by their direction angle. In the figure, the points are spaced at 45° intervals. From the expression for DC, we compute:

disturbance direction (°)	$T_h^{*'}$	$T_c^{*'}$	DC
0	0.5	0	$0.5B(F_{hr}\Delta T_h)$
45	0.5	0.5	$0.5B(F_{hr}\Delta T_h + F_{cr}\Delta T_c)$
90	0	0.5	$0.5B(F_{cr}\Delta T_c)$
135	-0.5	0.5	$0.5B(\lvert -F_{hr}\Delta T_h + F_{cr}\Delta T_c \rvert)$

and so forth. The constant B is

$$B = \left| \frac{1}{T_{hr} - T_{cr}} \right| \left(\frac{1}{\Delta F_h^{\ 2}} + \frac{1}{\Delta F_c^{\ 2}} \right)^{1/2} \tag{1-36}$$

As we surmised from examining the DC expression, the largest DC occurs at 45 and 225°.

we examine whether our MV are up to the job

DC tells us which combinations of disturbances require the largest adjustment of manipulated variables. At these conditions, we should examine just how large this adjustment is. For example, at 45°, the scaled cold flow manipulation from (1-34) is

$$F_{c,pc}^{*'} = \frac{0.5(F_{hr}\Delta T_h + F_{cr}\Delta T_c)}{(T_{hr} - T_{cr})\Delta F_c} \tag{1-37}$$

That is, we must increase the cold flow. To see how much flow is required, we write the flow in physical terms

$$F_{c,pc} = F_{cr} + F^{*'}_{c,pc}\Delta F_c \qquad (1\text{-}38)$$

and substituting for the scaled deviation variable, we find

$$F_{c,pc} = F_{cr} + \frac{0.5\left(F_{hr}\Delta T_h + F_{cr}\Delta T_c\right)}{\left(T_{hr} - T_{cr}\right)} \qquad (1\text{-}39)$$

If this value exceeds the value $F_{c.max}$ we had used in scaling F_c, we cannot overcome the 45° disturbance conditions with our present design. (At least, according to the linear system approximation to our original model, which is itself an approximation to the physical operation. We should pay attention to our models, but not believe them beyond their worth.)

DC doesn't depend on the pairing of CV and MV
DC is a consequence of the process model, that is, the dependence of the output variables on the disturbance and manipulated inputs. It assumes nothing about the way that control is actually conducted; it does assume that control has been effective, so that a disturbance input has been compensated by a manipulated variable input to return the output to set point at steady state. Actually designing the controllers to do this is another topic.

references

(1) T. Marlin, Process Control, 2nd ed., McGraw-Hill, 2000.
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nomenclature

C_p	liquid specific heat
f	a nonlinear function of several independent variables
F	volumetric flow rate
$\underline{\underline{P}}_d$	matrix of gain coefficients for the disturbance variables
$\underline{\underline{P}}_m$	matrix of gain coefficients for the manipulated variables
T	temperature
T_{ref}	thermodynamic reference temperature for enthalpy
V	volume of the piping system
x_1, x_2	independent variables
$\underline{x}_d$	vector of input variables into the system, the disturbance variables
$\underline{x}_m$	vector of input variables into the system, the manipulated variables
$\underline{y}$	vector of output variables from the system, the controlled variables
γ	ratio of hot supply reference flow to total reference flow
λ	element in the relative gain array
$\underline{\underline{\Lambda}}$	the relative gain array matrix
ρ	liquid density

abbreviations

CV	"controlled variable", a system output that we wish to maintain at a set point value

DC	"disturbance cost"
DV	"disturbance variable", a system input that we have no influence over
MV	"manipulated variable", a system input that we may adjust for our purposes
RGA	"relative gain array"

<u>subscripts</u>

c	cold water supply stream
h	hot water supply stream
max	maximum value of a variable
min	minimum value of a variable
pc	perfect control has been exerted on CV
r	a reference operating condition around which we derive a linear approximation

<u>superscripts</u>

$'$	indicates a deviation variable; i.e., the physical variable minus a reference value
*	indicates a scaled variable; i.e., variable has been divided by its operating range