Lecture #21: TA Tutorial on BVPs, FEMLAB®.

BVPs

What methods to use for each situation:

- 1D
  - Finite Differences
  - ODE's
    - 2nd order: → Two 1st order
    - 1st order: → Shooting
- 2D
  - Finite Differences
  - Method of Lines
    - Stiff
  - Non-uniform grid
- 3D
  - Finite Element
  - Finite Volume

  } Generally use commercial code

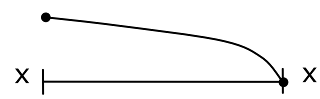


Figure 1. Boundary layer.

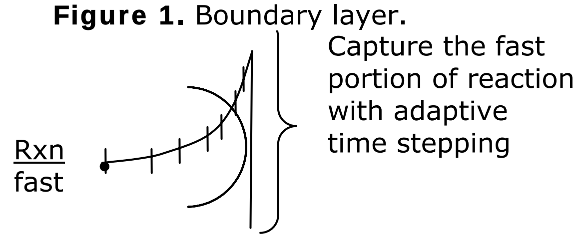


Figure 2. Adaptive time stepping.

Coding Boundary Conditions

Linear: $\underline{\underline{A}}\cdot\underline{x} = \underline{b}$

$\underline{x}$ → state variables at grid points

$$\begin{pmatrix} \ddots & \ddots & & & & \ddots & & 0 \\ \ddots & \ddots & \ddots & \ddots & & 0 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & & & \ddots \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ \ddots & 0 & & & \ddots & \ddots & \ddots & \ddots \\ & \ddots & & & & \ddots & \ddots & \ddots \\ 0 & & \ddots & & & & \ddots & \ddots \end{pmatrix} \begin{bmatrix} x_{1,1} \\ x_{i,j} \\ \vdots \\ x_{N_x,N_y} \end{bmatrix} = 0$$

BC: $x_{1,j} = 1$ $\quad \left.\frac{d\theta}{dx}\right|_{1,j} \cong -\left(\frac{3\theta_{1,j} - 4\theta_{2,j} - \theta_{3,j}}{2\Delta x}\right)$

$$1 \rightarrow \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} \begin{bmatrix} x_{1,1} \\ \vdots \\ x_{1,2} \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{4}{3} & \frac{1}{3} & \cdots \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ x_{3,1} \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \\ \end{bmatrix} \qquad \text{for } \left.\frac{d\theta}{dx}\right|_{1,j} = 0$$

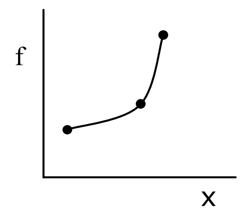
Nonlinear

use fsolve

BC: $x_{1,j} = 1 \longrightarrow x_{1,j} - 1 = 0$ {set to zero}

Non-Uniform Grid

Throw out all original equations



polynomial between three points

$$p(x) = \Sigma f_i L_i(x) \qquad L_i(x) = \prod_{\substack{k=0 \\ k\neq i}}^{N} \left[\frac{x - x_k}{x_i - x_k} \right]$$

Figure 3. A graph of a polynomial fit of three points.

$$f(x) = f_1 \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + f_2 \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + f_3 \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

$\left.\frac{df}{dx}\right|_{x_2}$: differentiate the above expression f(x) and evaluate at x_2.

$$\left.\frac{df}{dx}\right|_{x_2} = f_1 \frac{(x_2-x_3)}{(x_1-x_2)(x_1-x_3)} + f_2 \frac{2x_2-x_3-x_1}{(x_2-x_1)(x_2-x_3)} + f_3 \frac{(x_2-x_1)}{(x_3-x_1)(x_3-x_2)}$$

$$\left.\frac{d^2 f}{dx^2}\right|_{x_2} = \frac{2 f_1}{(x_1-x_2)(x_1-x_3)} + \frac{2 f_2}{(x_2-x_1)(x_2-x_3)} + \frac{2 f_3}{(x_3-x_1)(x_3-x_2)}$$

MatLAB Equation: *nonuniform_example*

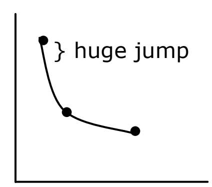


Figure 4. Graph of a function that is steep in the beginning.

Scaling

$$v_z \frac{\partial C_A}{\partial z} = D\left[\frac{\partial^2 C_A}{\partial z^2} + \frac{\partial^2 C_A}{\partial y^2}\right] + k(C_A C_B - C_{AB}/K)$$

$$Z = z/L$$

$$Y = y/b$$

$$\frac{\partial C_A}{\partial Z} = \left(\frac{DL}{v_z b^2}\right)\frac{\partial^2 C_A}{\partial Y^2} + \left(\frac{D}{v_z L}\right)\frac{\partial^2 C_A}{\partial Z^2} + \left(\frac{L}{v_z}\right)R$$

$$\frac{\partial C_A}{\partial Z} = 10^{-3}\frac{\partial^2 C_A}{\partial Y^2} \Rightarrow (dy)^2 = 10^{-3}\delta z$$

$$\text{so if } \delta z = 10^{-1},\ \text{choose } \delta y = 10^{-2}$$

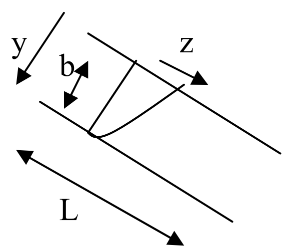


Figure 5. Diagram of pipe flow with reaction and diffusion.

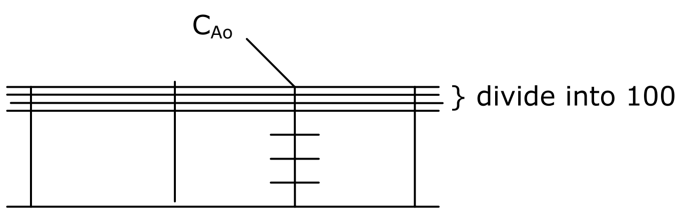


Figure 6. Non-uniform grid.

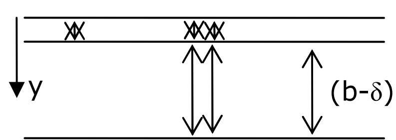


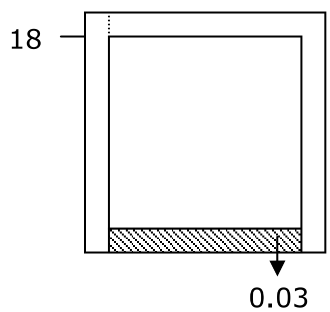
Figure 7. Division of problem into δ and $(b-\delta)$ regions.

$$\frac{\partial C_A}{\partial z} = 10^{-9}\frac{\partial^2 C_A}{\partial z^2}$$

Problem 1

Using FEMLAB®

* space dimension: axial symmetry 2D



axis/grid setting: $r_{min} = -0.01$

$r_{max} = 0.06$

$z_{min} = -1$

$z_{max} = 20$

Figure 8. Diagram of FEMLAB Example.

Subdomain settings:

r = 0.001(r-(r/0.05)^2)

Boundary settings:

mesh mode puts in finite elements.

Solve the problem.

DONE IN UNDER 3 MINUTES!!!