

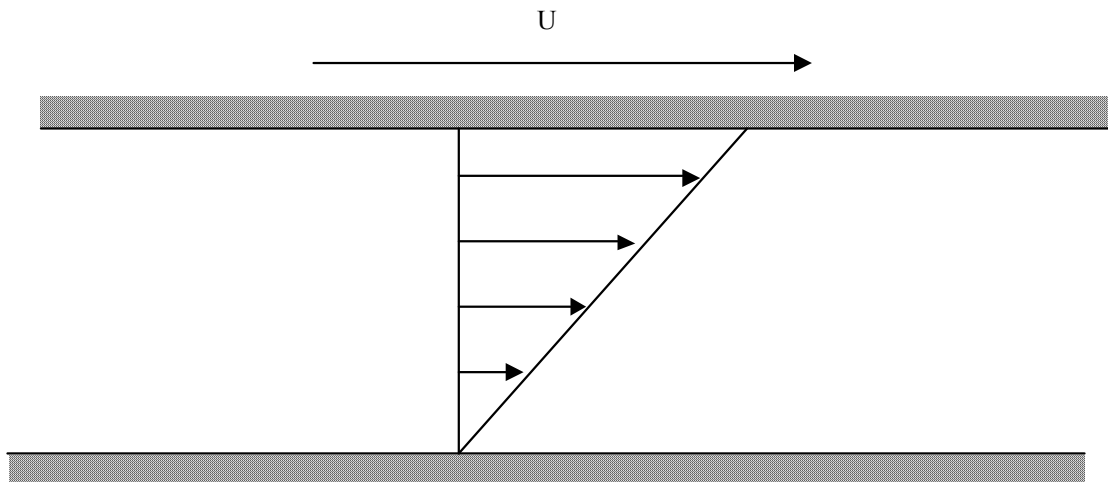
Derivation of the Viscous Flow Equations

To obtain the equation for viscous-dominated (inertial-free) flow, we need to start with the local force balance in the fluid, which is the same expression we used previously in a solid,

$$\frac{\partial \tau_{ij}}{\partial x_i} = 0 \quad (1)$$

where τ_{ij} is the stress tensor and the subscripts i, j denote the direction of the normal vector corresponding to the *surface* on which the stress acts and the *component* of the stress, respectively.

Next, we need to define an appropriate constitutive law for the fluid, analogous to Hooke's law for a solid. Here, however, an important distinction must be made. A solid, when subjected to a steady shear stress, deforms an amount proportional to the applied stress, then remains fixed in that deformed configuration. A fluid, on the other hand, when subjected to a steady shear stress, continues to deform, and the *rate* of shear deformation (rather than its magnitude) depends upon the level of applied stress. Hence, we need a constitutive law of a different form wherein the stresses are related to the rate of deformation rather than to the deformation itself.



For now, we restrict our attention to a *Newtonian fluid*, one in which the stress and rate of deformation are linearly related. We first need to define the rates of deformation, which, as might be expected, are related to the velocity field within the fluid, v_i . In the 1-2 plane, a simple shearing deformation is shown in Fig. 1 in which the

fluid, contained between two plates, is sheared by the action of a steady force per unit area τ_{21} to the right. If we take the bottom plate to be stationary ($u_1=0$), then the upper plate moves to the right at a speed U and the velocity profile in the fluid is found (e.g., by experiment) to be linear with x_2 . We define a Newtonian fluid as one in which the rate of deformation or the *strain rate*, $d\gamma/dt$, is linearly related to the applied stress so that

$$\tau_{21} \propto \frac{d\gamma}{dt} \quad (2)$$

From geometry, it can be seen that

$$\frac{d\gamma}{dt} = \frac{dv_1}{dx_2} \quad (3)$$

so that we can rewrite (2) as an equality by introducing the constant of proportionality which we call the *coefficient of viscosity*, μ ,

$$\tau_{21} = \mu \frac{dv_1}{dx_2} \quad (4)$$

For many common fluids (e.g., air, water) μ is independent of the shear rate γ , and is termed *Newtonian*. Fluids for which the viscosity varies with $d\gamma/dt$ are referred to as *non-Newtonian*. Generalizing eqn. (4) to three dimensions, we have:

$$\tau_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad \text{for } i \neq j \quad (5)$$

where you should note the strong similarity to Hooke's law for a solid where strain, rather than strain rate, is constitutively related to stress. It will become clear below why we have restricted this equation to cases $i \neq j$.

Before introducing this into eqn. (1), we need to take into account the fact that a uniform normal stress (i.e., a *hydrostatic pressure*) does not produce any motion in the fluid. We therefore define the *deviatoric stress* that essentially represents that stress with the hydrostatic component separated out. The deviatoric stress is denoted as τ' and is related to the pressure and total stress through the equation:

$$\tau_{ij} = -p\delta_{ij} + \tau'_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad (6)$$

The nine components of deviatoric stress comprise the *deviatoric stress tensor*. Note that the pressure p is defined as the mean of the three normal stresses:

$$p = \frac{\tau_{11} + \tau_{22} + \tau_{33}}{3} \quad (7)$$

Now, when we combine the overall force balance equation (1) with equation (6), we obtain:

$$\nabla p = \mu \nabla^2 \vec{V} \quad (8)$$

with $\vec{V} = v_1 \vec{i}_1 + v_2 \vec{i}_2 + v_3 \vec{i}_3$. This is often called the *Stokes Equation*.

When we use the Stokes Equation, we will typically make use of only one component of the vector equation. For example, we can write, in the x_i direction:

$$\frac{\partial p}{\partial x_i} = \mu \left(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2} \right) \quad (9)$$

Or, expressing (8) in cylindrical coordinates, the axial component *for axisymmetric flow*, is:

$$\frac{\partial p}{\partial z} = \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) \right] \quad (10)$$

where we have also assumed the flow to be *fully-developed*, i.e., not varying in the axial (z) direction.