Numerical Schemes for Scalar One-Dimensional Conservation Laws

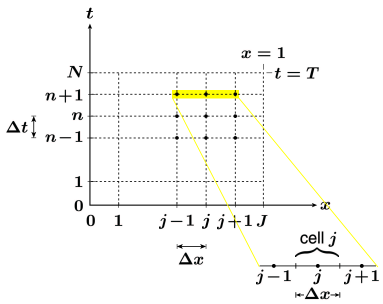
Lecture 12

1 Finite Volume Discretization

1.1 Computational Cells

SLIDE 1

$$x_j = j\Delta x$$
$$t^n = n\Delta t$$



1.2 Cell averages

SLIDE 2

Recall that in finite differences $\hat{u}_j^n \approx u(x_j, t^n)$.

We think of $\hat{u}_j^n$ as representing cell averages

$$\boxed{\hat{u}_j^n \approx \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t^n)\, dx}$$

This "new" interpretation can be easily extended to irregular grids.

2 Conservative Methods

2.1 Definition

SLIDE 3

Applying integral form of conservation law to a cell j

$$\frac{d}{dt} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u\, dx = -\left[f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t)) \right]$$

suggests

$$\frac{\hat{u}_j^{n+1} - \hat{u}_j^n}{\Delta t}\, \Delta x = -\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n \right)$$

$$\Rightarrow \qquad \boxed{\hat{u}_j^{n+1} = \hat{u}_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n \right)}$$

We consider here only explicit schemes, but implicit schemes are also possible.

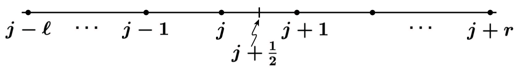
2.2 Numerical Flux function

SLIDE 4

$$F_{j+\frac{1}{2}} \equiv F\left(\hat{u}_{j-l}, \hat{u}_{j-l+1}, \ldots, \hat{u}_j, \ldots, \hat{u}_{j+r}\right)$$

and F is a **numerical flux function** of $l+r+1$ arguments that satisfies the following **consistency** condition

$$\boxed{F(u, u, \ldots, u, u) = f(u)}$$



We will sometimes omit the time superscript with the understanding that left and right hand sides are evaluated at the same time. Thus, the above flux function expression implies that

$$F^n_{j+\frac{1}{2}} \equiv F\left(\hat{u}^n_{j-l}, \hat{u}^n_{j-l+1}, \ldots, \hat{u}^n_j, \ldots, \hat{u}^n_{j+r}\right).$$

2.3 Lax-Wendroff Theorem

SLIDE 5

> If the solution of a **conservative** numerical scheme converges as $\Delta x \rightarrow 0$ with $\frac{\Delta t}{\Delta x}$ fixed, then it **converges to a weak solution** of the conservation law.

N1

$\Rightarrow$ **shock capturing** schemes are possible N2

Note 1 ***The Lax-Wendroff Theorem***

While the Lax-Wendroff theorem shows that **if** we converge to some solution as the grid is refined, then that solution will be a weak solution of the conservation law, it does not guarantee that we will converge. In fact the consistency to our integral form of the conservation law is guaranteed if we employ a conservative numerical scheme as defined above. We know that in order to obtain convergence we require some notion of stability. Because we are dealing with a non-linear problem the concepts of stability used until now are not applicable. At the end of this lecture we will give sufficient conditions for a scheme to be non-linearly stable and hence convergent.
The theorem also does not guarantee that the weak solution obtained satisfies the entropy condition. If more than one weak solution exists for a given problem, then different conservative numerical schemes may converge to different answers. We will discuss entropy-satisfying schemes later in the lecture.

Note 2	**Shock Capturing vs. Shock Fitting**

We say that a scheme captures shocks when the shocks or discontinuities appear in the solution as regions of large gradients without having to give them any special treatment. If we use conservative schemes, the Lax-Wendroff theorem guarantees that converge, if it occurs, will be to a weak solution. We know that weak solutions satisfy the jump conditions and therefore give the correct shock speed.
An alternative to shock capturing schemes are the so called shock fitting methods. In these methods, one needs to assume that a discontinuity will be present in the solution. The numerical algorithm iteratively determines the strength and speed of that discontinuity using the Rankine-Hugoniot jump relation. Shock fitting schemes will not be considered in this lectures. They are considered old and hardly used nowadays. The main disadvantage is that one requires a fair amount of knowledge about the solution before one actually computes it. They are also very difficult to extend to multidimensions where one can have very complex interactions involving several shock systems and consequently no a-priori knowledge about the structure of the solution.

2.3.1 Shock Capturing

SLIDE 6

In the exact problem:

$$\frac{d}{dt}\int_{x_0}^{x_J} u \; dx = -(f_0 - f_J)$$

Here $f_0 = f(u(x_0, t))$ *and* $f_J = f(u(x_J, t))$.

A conservative numerical scheme satisfies an analogous discrete condition: N3

$$\begin{aligned} \frac{\Delta x}{\Delta t}\sum_{j=0}^{J}(\hat{u}_j^{n+1} - \hat{u}_j^n) &= -\sum_{j=0}^{J}\left(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right) \\ &= -\left(F_{J+\frac{1}{2}} - F_{-\frac{1}{2}}\right) \end{aligned}$$

We see that due to the cancellation of all interior fluxes we are only left with the boundary fluxes. The form of these boundary fluxes will depend on the boundary conditions.

Note 3	**Discrete Conservation**

The basic priciple underlying a conservation law is that the total quantity of a conserved variable in any region changes only due to flux through the boundaries. We saw this in the last lecture when we derived some conservation laws

(conservation of mass, cars, ...). The expression given in the slide is an analogous discrete form of this principle. This discrete conservation means that any shocks computed by the conservative numerical scheme must be in the "correct" location. A non-conservative method can give a solution with the shock propagating at the wrong speed. This cannot happen with a conservative method, since an incorrect shock speed would lead to an incorrect flux, and thus conservation would not be preserved. The solution computed with a conservative method might not accurately resolve the shock (it may be smeared out), but when the grid is refined sufficiently, the discontinuitiy will be located in the correct position.
For example, consider a non-conservative upwind scheme for Burgers' equation:

$$\hat{u}_j^{n+1} = \hat{u}_j^n - \begin{cases} \frac{\Delta t}{\Delta x}\hat{u}_j^n \left(\hat{u}_j^n - \hat{u}_{j-1}^n\right) & \hat{u}_j^n > 0 \\ \frac{\Delta t}{\Delta x}\hat{u}_j^n \left(\hat{u}_{j+1}^n - \hat{u}_j^n\right) & \hat{u}_j^n < 0 \end{cases}$$

$$\begin{aligned} \frac{\Delta t}{\Delta x}\sum_{j=1}^{J} \Delta \hat{u}_j^n &= \sum_{\hat{u}_j > 0} -\hat{u}_j^n \left(\hat{u}_j^n - \hat{u}_{j-1}^n\right) + \sum_{\hat{u}_j < 0} -\hat{u}_j^n \left(\hat{u}_{j+1}^n - \hat{u}_j^n\right) \\ &= -\left(F_{J+\frac{1}{2}} - F_{-\frac{1}{2}}\right) + \textbf{conservation errors} \end{aligned}$$

If the solution is smooth, the conservation errors are $\mathcal{O}(\Delta x)$. If the solution is not smooth, the conservation errors are $\mathcal{O}(1)$.

2.4 First Order Upwind

2.4.1 Linear Advection Equation

SLIDE 7

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad a \text{ constant} > 0$$

$$\hat{u}_j^{n+1} = \hat{u}_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}}^{UP} - F_{j-\frac{1}{2}}^{UP}\right)$$

Let $\quad F_{j+\frac{1}{2}}^{UP} \equiv a\,\hat{u}_j \quad \left(F_{j-\frac{1}{2}}^{UP} = a\,\hat{u}_{j-1}\right)$

Note that for this definition of the numerical flux function the consistency condition is clearly statified. i.e. $F(u) = au = f(u)$.

$\Rightarrow$
$$\hat{u}_j^{n+1} = \hat{u}_j^n - \frac{\Delta t\, a}{\Delta x}(\hat{u}_j - \hat{u}_{j-1})$$

SLIDE 8

What about $a < 0$?
We can write,

$$\hat{u}_j^{n+1} = \hat{u}_j^n - \frac{a\Delta t}{\Delta x} \begin{cases} \hat{u}_j^n - \hat{u}_{j-1}^n & a > 0 \\ \hat{u}_{j+1}^n - \hat{u}_j^n & a < 0 \end{cases}$$

or

$$\hat{u}_j^{n+1} = \hat{u}_j^n - \frac{a\Delta t}{2\Delta x} \left(\hat{u}_{j+1}^n - \hat{u}_{j-1}^n\right) + \frac{|a|\Delta t}{2\Delta x} \left(\hat{u}_{j+1}^n - 2\hat{u}_j^n + \hat{u}_{j-1}^n\right)$$

Note that by introducing the absolute value we are able to write a single expression that takes into account the dependency of the difference stencil on the sign of a.

SLIDE 9

In **conservative form**:

$$\hat{u}_j^{n+1} = \hat{u}_j^n - \frac{\Delta t}{\Delta x} \left(F_{j+\frac{1}{2}}^{UPn} - F_{j-\frac{1}{2}}^{UPn}\right)$$

$$\boxed{F_{j+\frac{1}{2}}^{UP} = \frac{1}{2}a(\hat{u}_{j+1} + \hat{u}_j) - \frac{1}{2}|a|(\hat{u}_{j+1} - \hat{u}_j)}$$

$$\begin{aligned} F_{j+\frac{1}{2}}^{UP} &= a\hat{u}_j & a > 0 \\ F_{j+\frac{1}{2}}^{UP} &= a\hat{u}_{j+1} & a < 0 \end{aligned}$$

We see that although the first order upwind method was originally derived using finite difference, and characteristic interpolation arguments, it can also be interpreted as a conservative finite volume scheme were we solve for cell solution averages rather than pointwise values.

We note that the upwind scheme written in this manner is precisely a FCTS scheme (forward in time centered in space) with the explicit addition of a second difference term. As we know from the linear analysis this term is required for stability.

2.4.2 Nonlinear Case

SLIDE 10

In the nonlinear case,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

the flux becomes N4

$$\boxed{F_{j+\frac{1}{2}}^{UP} = \frac{1}{2}\left(\hat{f}_{j+1} + \hat{f}_j\right) - \frac{1}{2}|\hat{a}_{j+\frac{1}{2}}|\left(\hat{u}_{j+1} - \hat{u}_j\right)}$$

$$\hat{a}_{j+\frac{1}{2}} = \begin{cases} \frac{\hat{f}_{j+1}-\hat{f}_j}{\hat{u}_{j+1}-\hat{u}_j} & \text{if} \quad \hat{u}_{j+1} \neq \hat{u}_j \\ f'(\hat{u}_j) & \text{if} \quad \hat{u}_{j+1} = \hat{u}_j \end{cases}$$

Here $\hat{f}_j$ denotes $f(\hat{u}_j)$. The above choice of $\hat{a}$ guarantees that one sided approximation is obtained, i.e.

$$F_{j+\frac{1}{2}} = \begin{cases} \hat{f}_j & \hat{a}_{j+\frac{1}{2}} > 0 \\ \hat{f}_{j+1} & \hat{a}_{j+\frac{1}{2}} < 0 \end{cases}$$

Note 4	***First Order Upwind Scheme***

The first order upwind scheme is conservative, and for Δt sufficiently small, it can be shown to be convergent (later on in this lecture we will discuss the requirements for convergence). The Lax-Wendroff theorem therefore ensures that it will converge to a weak solution.

We show below that the Lax-Wendroff and Beam-Warming algorithms are also conservative schemes and therefore admit a finite volume interpretation.

2.5 Lax-Wendroff

SLIDE 11

$$\boxed{F^{LW}_{j+\frac{1}{2}} = \frac{1}{2}\left(\hat{f}_{j+1} + \hat{f}_j\right) - \frac{1}{2}\,\hat{a}^2_{j+\frac{1}{2}}\,\frac{\Delta t}{\Delta x}\,(\hat{u}_{j+1} - \hat{u}_j)}$$

$\hat{a}_{j+\frac{1}{2}}$ *is again defined as*

$$\hat{a}_{j+\frac{1}{2}} = \begin{cases} \frac{\hat{f}_{j+1}-\hat{f}_j}{\hat{u}_{j+1}-\hat{u}_j} & \textit{if} \quad \hat{u}_{j+1} \neq \hat{u}_j \\ f'(\hat{u}_j) & \textit{if} \quad \hat{u}_{j+1} = \hat{u}_j \end{cases}$$

For the linear equation

$$\hat{u}^{n+1}_j = \hat{u}_j - \frac{C}{2}\left(\hat{u}^n_{j+1} - \hat{u}^n_{j-1}\right) + \frac{C^2}{2}\left(\hat{u}^n_{j+1} - 2\hat{u}^n_j + \hat{u}^n_{j-1}\right)$$

$$\boxed{C = a\Delta x/\Delta t}$$

2.6 Beam-Warming

SLIDE 12

$$
\boxed{
\begin{aligned}
F^{BW}_{j+\frac{1}{2}} &= \tfrac{1}{4}\left(-\hat{f}_{j+2} + 3\hat{f}_{j+1} + 3\hat{f}_j - \hat{f}_{j-1}\right) - \hat{a}^2_{j+\frac{1}{2}} \tfrac{\Delta x}{4\Delta t}\left(\hat{u}_{j+2} - \hat{u}_{j+1} + \hat{u}_j - \hat{u}_{j-1}\right) \\
&- \tfrac{s_{j+\frac{1}{2}}}{4}\left(-\hat{f}_{j+2} + 3\hat{f}_{j+1} - 3\hat{f}_j + \hat{f}_{j-1}\right) + s_{j+\frac{1}{2}}\, a^2_{j+\frac{1}{2}} \tfrac{\Delta t}{4\Delta x}\left(\hat{u}_{j+2} - \hat{u}_{j+1} - \hat{u}_j + \hat{u}_{j-1}\right)
\end{aligned}
}
$$

$$\boxed{s_{j+\frac{1}{2}} = a_{j+\frac{1}{2}}/|a_{j+\frac{1}{2}}|}$$

For the linear equation

$$
\begin{aligned}
\hat{u}^{n+1}_j &= \hat{u}^n_j - \frac{C}{2}\left(3\hat{u}^n_j - 4\hat{u}^n_{j-1} + \hat{u}^n_{j-2}\right) + \frac{C^2}{2}\left(\hat{u}^n_j - 2\hat{u}^n_{j-1} + \hat{u}^n_{j-2}\right) \qquad a > 0 \\
\hat{u}^{n+1}_j &= \hat{u}^n_j - \frac{C}{2}\left(-3\hat{u}^n_j + 4\hat{u}^n_{j+1} - \hat{u}^n_{j+2}\right) + \frac{C^2}{2}\left(\hat{u}^n_{j+2} - 2\hat{u}^n_{j+1} + \hat{u}^n_j\right) \qquad a < 0
\end{aligned}
$$

2.7 Entropy Solutions

SLIDE 13

Do these schemes converge to the entropy satisfying solution?

EXAMPLE:
Consider a non-physical solution to Burgers' equation:

$$u(x,t) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

$$\text{i.e. } \hat{u}^n_j \text{ is either } 1 \text{ or } -1 \quad \Rightarrow \quad f_j = \tfrac{1}{2} \quad \forall j$$

2.7.1 Example

SLIDE 14

First order upwind:

$$F^{UP}_{j+\frac{1}{2}} = \frac{1}{2}\left(\hat{f}_{j+1} + \hat{f}_j\right) - \frac{1}{2}|\hat{a}_{j+\frac{1}{2}}|\left(\hat{u}_{j+1} - \hat{u}_j\right)$$

Since either $\hat{a}_{j+\frac{1}{2}}$ or $\hat{u}_{j+1} - \hat{u}_j$ is zero $\forall j$ *Because either* $\hat{f}_j = \hat{f}_{j+1}$, *or* $\hat{u}_j = \hat{u}_{j+1}$.

$$\Rightarrow \quad F^{UP}_{j+\frac{1}{2}} = \tfrac{1}{2} \quad \forall j \Rightarrow \quad F^{UP}_{j+\frac{1}{2}} - F^{UP}_{j-\frac{1}{2}} = 0 \quad \forall j$$

$$\Rightarrow \quad \hat{u}^{n+1}_j = \hat{u}^n_j$$

The entropy-violating solution is preserved N5

Note 5 ***Entropy Satisfying Solutions***

It turns out that the first order upwind, the Lax-Wendroff and the Beam-Warming schemes allow for entropy violating solutions. These schemes cannot distinguish between shocks and expansions.
To determine in advance if a general numerical scheme will only produce entropy satisfying solutions is very difficult. One possible approach is to derive an

entropy function for our discrete scheme and prove a corresponding **entropy cell inequality** of the form

$$\frac{U(u_j^{n+1}) - U(u_j^n)}{\Delta t} + \frac{H_{j+\frac{1}{2}}^n - H_{j-\frac{1}{2}}^n}{\Delta x} \leq 0,$$

where $H_{j+\frac{1}{2}}^n$ is the numerical entropy flux associated with U.
Verifying an entropy cell inequality is not easy to do in general. There are certain classes of schemes, as we shall see below, which can be shown to guarantee entropy satisfying solutions. It is often much simpler to verify whether a scheme belong to one such classes.

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3 Entropy Satisfying Schemes

3.1 Monotone Schemes

SLIDE 15

If a scheme can be written in the form

$$\hat{u}_j^{n+1} = H\left(\hat{u}_{j-l}^n, \hat{u}_{j-l+1}^n, \ldots, \hat{u}_j^n, \ldots, \hat{u}_{j+r}^n\right)$$

with $\dfrac{\partial H}{\partial u_i} \geq 0 \qquad i = j-l, \ldots, j, \ldots, j+r,$

then the scheme is **monotone** and is

- **entropy satisfying**
- **at most first order accurate**

N6

Note 6 ***Convergence in the presence of Discontinuities***

For discontinuous solutions we seek convergence in weaker norms that those we have used for problems with smooth solutions. Most of the convergence results available for finite volume methods for conservation laws measure convergence in the $p = 1$ norm. This norm can be seen to be weaker than the $p = 2$ and $p = \infty$ in the sense that it is possible to find a scheme that will converge in the $p = 1$ norm, but will not converge in the other norms. In fact, shock capturing methods are not convergent in the $p = \infty$ norm when the solution is discontinuous.
Moreover, the so called "first order schemes" i.e. schemes that have a truncation error $\mathcal{O}(\Delta x, \Delta t)$, often converge at an even lower rate in the presence of discontinuities [L]. Note that our definition of truncation error is based on the assumption that the exact solution is smooth and therefore is not applicable in the discontinuous case.

3.1.1 Godunov's Method

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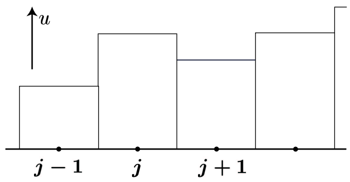
The best know monotone scheme is Godunov's method



Assume piecewise constant solution over each cell. Compute interface flux by **solving interface** (Riemann) **problem exactly**.

SLIDE 17

$$
\begin{aligned}
F^{Gn}_{j+\frac{1}{2}} &= f\left(u(x_{j+\frac{1}{2}}, t^{n+})\right) \\
&= \begin{cases} \displaystyle\min_{u\in[u_j,u_{j+1}]} f(u) & u_j < u_{j+1} \\ \displaystyle\max_{u\in[u_j,u_{j+1}]} f(u) & u_j > u_{j+1} \end{cases}
\end{aligned}
$$

Then,

$$
\hat{u}_j^{n+1} = \hat{u}_j^n - \frac{\Delta t}{\Delta x}\left(F^{Gn}_{j+\frac{1}{2}} - F^{Gn}_{j-\frac{1}{2}}\right)
$$

The above expression gives the exact flux for the Riemann problem and is valid for any scalar conservation law, with convex as well as non-convex fluxes. In addition, it gives the correct flux corresponding to the weak solution satisfying Oleinik's entropy condition. We point out that this flux is only valid for a short time. In fact, it is valid until the waves generated from the solution of one Riemann problem start interacting with the waves generated by neighboring interfaces. The fact that the solution is exact for short times is due to the particular form of the solutions of the Riemann problem (i.e although the general solution of the equations is a function of x and t, the solution to the Riemann problem can be expressed as a function of a single varibale, namely x/t; this property is known as similarity; see [L] for details).

SLIDE 18

Applied to Burgers' equation

$$
F^{G}_{j+\frac{1}{2}} = \begin{cases}
\frac{1}{2}\hat{u}^2_{j+1} & \hat{u}_j, \hat{u}_{j+1} < 0 & \\
\frac{1}{2}\hat{u}^2_{j} & \hat{u}_j, \hat{u}_{j+1} > 0 & \\
0 & \hat{u}_j < 0 < \hat{u}_{j+1} & \text{(expansion)} \\
\frac{1}{2}\hat{u}^2_{j} & \hat{u}_j > 0 > \hat{u}_{j+1} & \frac{1}{2}(\hat{u}_{j+1} + \hat{u}_j) > 0 \\
\frac{1}{2}\hat{u}^2_{j+1} & \hat{u}_j > 0 > \hat{u}_{j+1} & \frac{1}{2}(\hat{u}_{j+1} + \hat{u}_j) < 0
\end{cases}
$$

We note that this scheme produces a flux which is very similar to that of the upwind scheme but with the essential difference that the numerical flux is equal to zero in the expansion case. This, apparently, minor modification allows the scheme to distinguish between shocks and rarefactions, and thus produce an approximation to the proper entropy satisfying solution.

The main drawback of monotone schemes is their low accuracy. In order to develop schemes we need to look at a wider class of methods. One such class is formed by the so called E-schemes.

3.2 E-Schemes

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If the numerical flux $F_{j+\frac{1}{2}}$ satisfies

$$\boxed{\text{sign}(\hat{u}^n_{j+1} - \hat{u}^n_j)(F^n_{j+\frac{1}{2}} - f(u)) \leq 0 \quad \forall u \in [\hat{u}_j, \hat{u}_{j+1}]}$$

An E-scheme is

- **entropy satisfying**
- **at most first order accurate**

N7

Note 7 — ***E-Schemes***

All monotone schemes are E-schemes (not vice-versa). E-schemes include "fixes" to non-entropy satisfying schemes.
An example of such scheme is the scheme that can be constructed by writing the first order upwind scheme as $\hat{u}^{n+1}_j = \hat{u}^n_j - \frac{\Delta t}{\Delta x}(\hat{f}_j - \hat{f}_{j-1})$ (assume $\hat{a} > 0$), and splitting the flux $\hat{f}_j - \hat{f}_{j-1}$, into $\hat{f}_j - \hat{f}_s$ and $\hat{f}_s - \hat{f}_{j-1}$ whenever f' changes sign between $\hat{u}_j$ and $\hat{u}_{j-1}$, and $\hat{f}_s$ is the value of the flux at the sonic point; i.e. the point where f' is zero. The two parts are then added to either end of the cell instead of all to one end.
Entropy-satisfying schemes can be used as building blocks on which to build higher order schemes with reasonable confidence of obtaining the correct physical solution.

3.3 Summary

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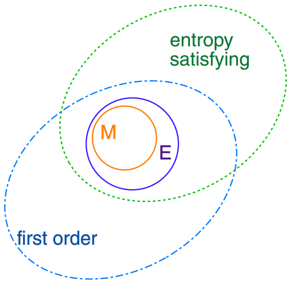

We see that we can construct some schemes that will guarantee entropy satisfying solutions e.g. monotone, E-schemes. The main drawback with these schemes is that they are very inaccurate. In practical applications, specially in realistic multidimensional computations typically we can not afford the number of grid points required by the first order schemes to produce accurate answers.

4 TVD Methods

4.1 Motivation

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First order schemes give poor resolution but can be made to produce entropy satisfying and **non-oscillatory solutions**

(e.g.monotone and E-schemes).

Higher order schemes (at least the ones we have seen so far) produce non-entropy satisfying and **oscillatory solutions**.

Oscillations are generated at discontinuities (where the discrete solution contains high frequency components) and are due to dispersion errors (recall that dispersion errors are much larger the higher the wavenumber). These oscillations can lead to non-linear instability.

We will look for yet another class of schemes referred to as TVD (Total Variation Diminishing) in an attempt to produce oscillation free solutions and high accuracy. TVD methods and its variants represent the state of the art in numerical methods for solving problems which involve shock waves.

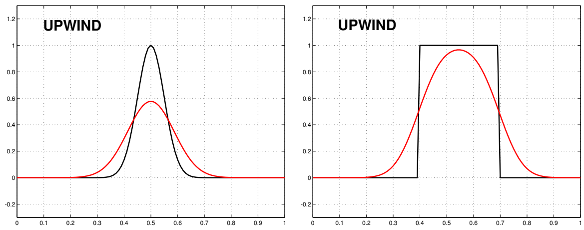
Good criterion to design "high order" oscillation free schemes is based on the **Total Variation** of the solution.

Below we show some numerical results for the linear advection equation using the first order upwind method and the second order Lax-Wendroff method. We point *out that for the linear problem, the first order upwing method and Godunov's methods become identical. In fact, for the linear problem the solution is always defined by the initial data (i.e. characteristics are parallel) and the issue of entropy violating solutions does not arise.*

The linear advection equation with periodic boundary conditions is solved in the unit interval. The domain is subdivided into 100 *equal subintervals. In the figures we show the exact (initial) solution and the computed solution after* 200 *timesteps at a Courant number of 0.5. Two initial conditions with smooth (left) and discontinuous (right) data are considered.*

4.2 First Order Upwind

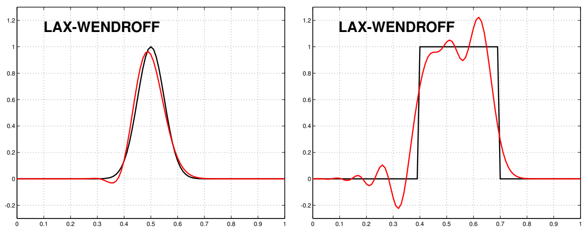
SLIDE 22



$$J = 100,\ \Delta x = 1/100,\ C = 0.5,\ N = 200$$

4.3 Lax-Wendroff

SLIDE 23



$$J = 100,\ \Delta x = 1/100,\ C = 0.5,\ N = 200$$

4.4 Definition

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Total Variation of the discrete solution

$$TV(\underline{\hat{u}}^n) = \sum_j \left| \hat{u}^n_{j+1} - \hat{u}^n_j \right|$$

We see that in the discrete case, the total variation is simply the sum of the absolute values of the differences between neighboring nodes.

If new extrema are generated $TV(\hat{\underline{u}})$ will increase.

We know that the total variation of the exact solution is non-increasing. Therefore we can define the class of Total Variation Diminishing (TVD) schemes as the class of schemes for which

$$\boxed{TV(\hat{\underline{u}}^{n+1}) \leq TV(\hat{\underline{u}}^{n})}$$

Total Variation Diminishing Schemes

We see clearly from the previous example that the Law-Wendroff is not TVD. It can be shown that Beam-Warming is not TVD either.

4.5 Some Properties

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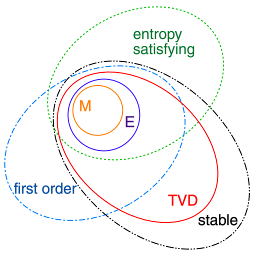
- All E-Schemes are TVD
- Conservative TVD Schemes
  ⇒ **Converge** to weak solutions



Unfortunately, not all TVD schemes are entropy satisfying. In practice however, TVD schemes which are built by modifying an E-scheme tend to produce entropy satisfying solutions. Although in most cases this can not be proven a-priori.

N8

Note 8 ***TVD Schemes and Stability***

The Lax-Wendroff theorem presented earlier does not say anything about whether the numerical scheme converges. It only guarantees that if it converges it will converge to a weak solution. To guarantee convergence, we require some form of stability. Unfortunately, the Lax equivalence theorem, which relies heavily on linearity, no longer applies in this case. As it turns out, the total variation diminishing property of a scheme can be used as a non-linear stability condition and it can be shown that togehter with consistency it guarantees the convergence of the numerical scheme.

To prove convergence of a non-linear method we use the concept of compactness. Essentially, we say that a normed space is compact when any infinite sequence of elements of that space, contains subsequences which converge to an element of that space. For finite dimensional spaces a closed, bounded set is compact. For inifinite dimensional spaces the situation is more complicated. However, it can be shown that the set of functions of bounded variation (TVB) with bounded support is compact. The fact that compactness guarantees the existence of convergent subsequences, combined with the Lax-Wendroff theorem gives us a convergence proof.
We note that convergence here is defined in the $p = 1$ norm, and that it can be to any weak solution. Since the weak solutions may be non-unique, convergence means that the distance between the computed solution and *any* weak solution tends to zero when the mesh parameters tend to zero. In particular, as we refine the mesh, different meshes may be closer to different weak solutions.

4.6 Conditions for TVD schemes

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If a scheme is written in the form

$$\hat{u}_j^{n+1} = \hat{u}_j^n + D_{j+\frac{1}{2}} \Delta \hat{u}_{j+\frac{1}{2}}^n - C_{j-\frac{1}{2}} \Delta \hat{u}_{j-\frac{1}{2}}^n$$

$$\boxed{\Delta \hat{u}_{j+\frac{1}{2}} = \hat{u}_{j+1} - \hat{u}_j}$$

We note that the coefficients $D_{j+\frac{1}{2}}$ and $C_{j-\frac{1}{2}}$, may depend on the $\hat{u}_j$'s.

it is TVD iff

$$\boxed{\begin{array}{cc} C_{j+\frac{1}{2}} & \geq 0 \\ D_{j+\frac{1}{2}} & \geq 0 \\ C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} & \leq 1 \end{array}}$$

Note 9 ***TVD Conditions: Proof***

Here we assume that the domain is either infinite or periodic.

$$\begin{aligned} TV(\underline{u}^{n+1}) &= \sum_j \left| \hat{u}_{j+1}^{n+1} - \hat{u}_j^{n+1} \right| \\ &= \sum_j \left| \Delta \hat{u}_{j+\frac{1}{2}}^n + D_{j+\frac{3}{2}} \Delta \hat{u}_{j+\frac{3}{2}}^n - C_{j+\frac{1}{2}} \Delta \hat{u}_{j+\frac{1}{2}}^n - D_{j+\frac{1}{2}} \Delta \hat{u}_{j+\frac{1}{2}}^n + C_{j-\frac{1}{2}} \Delta \hat{u}_{j-\frac{1}{2}}^n \right| \\ &\leq \sum_j \left| D_{j+\frac{3}{2}} \Delta \hat{u}_{j+\frac{3}{2}}^n \right| + \left| \left(1 - C_{j+\frac{1}{2}} - D_{j+\frac{1}{2}}\right) \Delta \hat{u}_{j+\frac{1}{2}}^n \right| + \left| C_{j-\frac{1}{2}} \Delta \hat{u}_{j-\frac{1}{2}}^n \right| \end{aligned}$$

$$= \sum_j \left|D_{j+\frac{1}{2}}\right| \left|\Delta \hat{u}^n_{j+\frac{1}{2}}\right| + \left|1 - C_{j+\frac{1}{2}} - D_{j+\frac{1}{2}}\right| \left|\Delta \hat{u}^n_{j+\frac{1}{2}}\right| + \left|C_{j+\frac{1}{2}}\right| \left|\Delta \hat{u}^n_{j+\frac{1}{2}}\right|$$

$$\leq TV(\underline{u}^n) \quad \boxed{\text{if} \quad \left|D_{j+\frac{1}{2}}\right| + \left|1 - C_{j+\frac{1}{2}} - D_{j+\frac{1}{2}}\right| + \left|C_{j+\frac{1}{2}}\right| \leq 1}$$

We note that the above inequalities can be made equalities for particular choices of the data $\hat{u}_j$. This means that the above conditions are necessary and sufficient if the scheme is to be TVD for all data.

4.6.1 Example: Upwind

SLIDE 27

Upwind scheme for linear equation, $a > 0$:

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$$

$$C_{j-\frac{1}{2}} = \tfrac{a\Delta t}{\Delta x}; \qquad D_{j+\frac{1}{2}} = 0$$

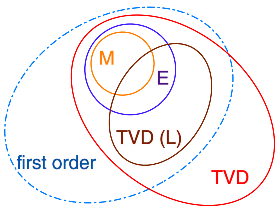
$$C_{j-\frac{1}{2}} = \frac{a\Delta t}{\Delta x} \leq 1$$

Stability-like condition !

4.7 Godunov's Theorem

SLIDE 28

No second or higher order accurate constant coefficient (**linear**) scheme can be **TVD**.



$\Rightarrow$ **Higher order TVD schemes must be nonlinear**

Note 10 ***Nonlinear TVD schemes***

We say that a scheme is liner if, when applied to a linear equation, produces a linear relation between the unknonws. i.e.

$$\hat{u}_j^{n+1} = \sum_k c_k u_{j-k}^n .$$

For a linear scheme the c_k's are constant, whereas for a nonlinear scheme, the c_k's depend on the values of u.

4.8 High Resolution Schemes

SLIDE 29

Consider the linear equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad a > 0$$

First order upwind (Godunov) scheme is

$$\hat{u}_j^{n+1} = \hat{u}_j^n - C\left(\hat{u}_j^n - \hat{u}_{j-1}^n\right)$$

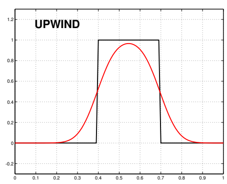
$$C = \frac{a\Delta t}{\Delta x}$$

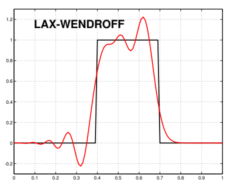
Oscillation free but smeared solutions.

SLIDE 30

Lax-Wendroff $\hat{u}_j^{n+1} = \hat{u}_j^n - \frac{C}{2}\left(\hat{u}_{j+1}^n - \hat{u}_{j-1}^n\right) + \frac{C^2}{2}\left(\hat{u}_{j+1}^n - 2\hat{u}_j^n + \hat{u}_{j-1}^n\right)$

Suffers from oscillations.





4.8.1 Anti-diffusion

SLIDE 31

Re-write the Lax-Wendroff scheme :

$$\hat{u}_j^{n+1} = \underbrace{\hat{u}_j^n - C\left(\hat{u}_j^n - \hat{u}_{j-1}^n\right)}_{\text{first order upwind}} - \underbrace{\frac{1}{2}C(1-C)\left(\hat{u}_{j+1}^n - 2\hat{u}_j^n + \hat{u}_{j-1}^n\right)}_{\text{anti-diffusive flux}}$$

$$F_{j+\frac{1}{2}}^{LW} = a\hat{u}_j + \frac{a}{2}(1-C)\left(\hat{u}_{j+1} - \hat{u}_j\right)$$

We note that au_j is simply $F_{j+\frac{1}{2}}^{UP}$. In order to obtain a TVD scheme, we must limit the amount of anti-diffusive flux.

Introduce **flux limiter** $\phi_{j+\frac{1}{2}}$:

$$F_{j+\frac{1}{2}}^{TVD} = a\hat{u}_j + \frac{a}{2}(1-C)\,\boldsymbol{\phi_{j+\frac{1}{2}}}\left(\hat{u}_{j+1} - \hat{u}_j\right)$$

4.8.2 Flux Limiters

SLIDE 32

$$\begin{aligned}\hat{u}_j^{n+1} &= \hat{u}_j^n - C\left(\hat{u}_j^n - \hat{u}_{j-1}^n\right)\\ &- \frac{1}{2}C(1-C)\left[\boldsymbol{\phi_{j+\frac{1}{2}}}\left(\hat{u}_{j+1}^n - \hat{u}_j^n\right) - \boldsymbol{\phi_{j-\frac{1}{2}}}\left(\hat{u}_j^n - \hat{u}_{j-1}^n\right)\right]\end{aligned}$$

It is essential that in order to preserve the conservative form of the scheme, the limiter is applied to the fluxes. At first sight it would appear as though one could limit the antidiffusion term added to each node directly. Note that this would clearly violate conservation.

If $\phi_j = \phi_{j-1} = 1 \Rightarrow$ Lax-Wendroff (not TVD)
If $\phi_j = \phi_{j-1} = 0 \Rightarrow$ Upwind (TVD)

Choose the limiter **as close as possible to** 1 **but enforcing TVD conditions**

SLIDE 33

Re-write

$$\begin{aligned}\hat{u}_j^{n+1} &= \hat{u}_j^n - C\Delta\hat{u}_{j-\frac{1}{2}} - \frac{1}{2}C(1-C)(\boldsymbol{\phi_{j+\frac{1}{2}}}\Delta\hat{u}_{j+\frac{1}{2}} - \boldsymbol{\phi_{j-\frac{1}{2}}}\Delta\hat{u}_{j-\frac{1}{2}})\\ &= u_j^n - C\left\{1 + \frac{1}{2}(1-C)\left[\frac{\phi_{j+\frac{1}{2}}}{r_{j+\frac{1}{2}}} - \phi_{j-\frac{1}{2}}\right]\right\}\Delta\hat{u}_{j-\frac{1}{2}}\end{aligned}$$

$$\boxed{r_{j+\frac{1}{2}} = \Delta\hat{u}_{j-\frac{1}{2}}/\Delta\hat{u}_{j+\frac{1}{2}}}$$

Recall the TVD test:

$$\hat{u}_j^{n+1} = \hat{u}_j^n + D_{j+\frac{1}{2}}\Delta\hat{u}_{j+\frac{1}{2}}^n - C_{j-\frac{1}{2}}\Delta\hat{u}_{j-\frac{1}{2}}^n$$

SLIDE 34

Take

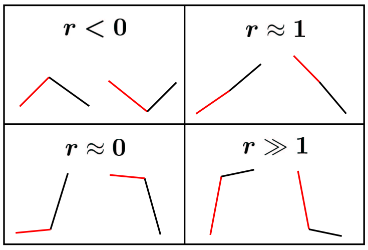
$$\begin{aligned}C_{j+\frac{1}{2}} &= C\left\{1 + \frac{1}{2}(1-C)\left[\frac{\phi_{j+\frac{1}{2}}}{r_{j+\frac{1}{2}}} - \phi_{j-\frac{1}{2}}\right]\right\}\\ D_{j+\frac{1}{2}} &= 0\end{aligned}$$

$$\text{TVD criterion} \Rightarrow \quad 0 \le C_{j+\frac{1}{2}} \le 1$$

4.8.3 Smoothness Monitor

SLIDE 35

Choose $\phi_{j+\frac{1}{2}}$ to be function of $r_{j+\frac{1}{2}}$



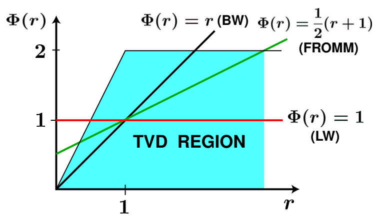
The ratio of consecutive slopes r gives an indication of the local smoothness of the discrete solution. When $r < 0$ we are in the presence of an extremum and we expect that anything different from $\phi = 0$ will produce oscillations. On the other hand when $r \approx 1$ we would expect the algorithm to be second order i.e. $\phi \approx 1$.

4.8.4 TVD region

SLIDE 36

It can be seen that the above TVD conditions are satisfied if

$$
\begin{aligned}
\phi(r) = 0 \qquad r \leq 0 \\
0 \leq \frac{\phi(r)}{r} \leq 2 \\
0 \leq \phi(r) \leq 2
\end{aligned}
$$



In the figure we show the choice of the limiters that would produce the Lax-Wendroff, Beam-Warming and Fromm's scheme (which is obtained as the average of the previous two). It can be seen that none of the limiter functions lies entirely in he TVD region, Lax-Wendroff being outside for small r, i.e. behind shocks, and Beam-Warming being outside for large r i.e. in front of shocks.

Since we are interested in second order schemes we further restict the region by taking it to be that part which is the convex average of the Lax-Wendroff and Beam-Warming schemes.

4.8.5 2nd Order TVD Region

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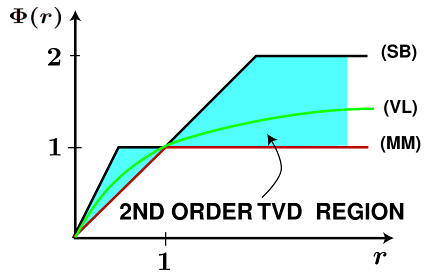

The figure shows some popular choices for the limiter function $\phi(r)$.

4.8.6 Popular Choices

SLIDE 38

Minmod $\phi(r) = \max(0, \min(1, r))$ *This is the most "difussive" limiter. It corresponds to the lower boundary of the TVD region.*

Superbee $\phi(r) = \max(0, \min(2r, 1), \min(r, 2))$ *This is the least "difussive" limiter. It corresponds to the upper boundary of the TVD region.*

Van Leer $\phi(r) = \dfrac{r + |r|}{1 + |r|}$

All produce **second order** schemes when the solution is smooth, and reduce to **upwind** at **discontinuities**.

All the limiters above possess the following symmetry property: $\frac{\phi(r)}{r} = \phi\left(\frac{1}{r}\right)$. This property ensures that the top corner of a discontinuity is treated symmetrically to a bottom corner.

4.8.7 Examples

Below we show some numerical results, for the same problem considered previously, using different numerical schemes. The first two schemes are linear second order and therefore not TVD. It should be pointed out that within this class Fromm's scheme does extremely well. In fact, for smooth data is arguably the best scheme amongst those presented. The last three schemes are TVD and correspond to the minmod, van Leer and superbee limiters.

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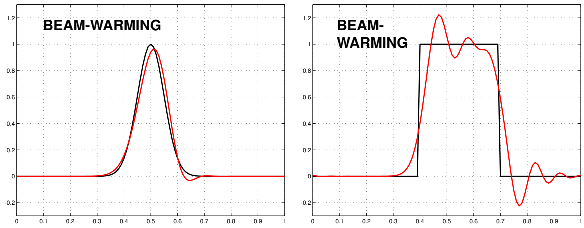
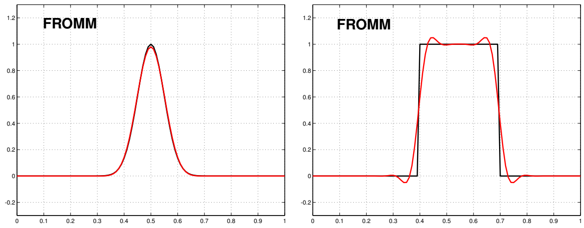


SLIDE 40



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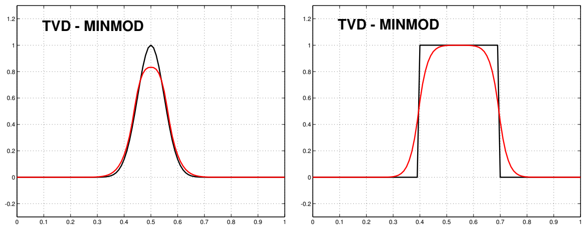

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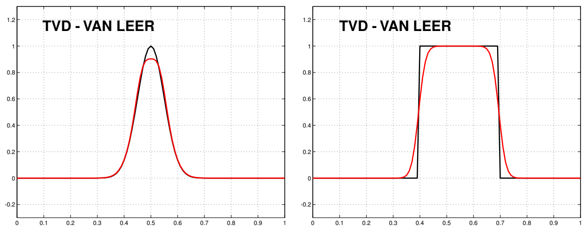

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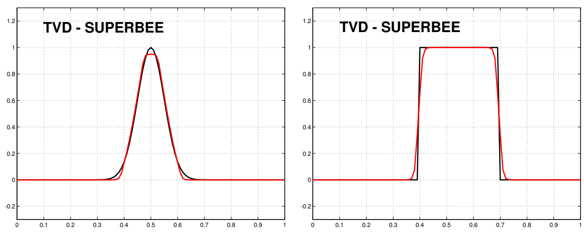

4.8.8 Non-linear extension

SLIDE 44

For a non-linear conservation law the formulation of flux limiters is extended to allow both positive and negative wave speeds

Note 11	*Nonlinear Extension*

Below we just give the expressions required to extend the above high resolution schemes to the general non-linear case. It can be verified that, for the linear problem, the expressions below reduce to the ones give above if the upwind scheme is taken to be the low order scheme.

$$
\begin{aligned}
\hat{u}_j^{n+1} \quad = \quad & \hat{u}_j^n - \frac{\Delta t}{\Delta x}\left(F^E_{j+\frac{1}{2}} - F^E_{j-\frac{1}{2}}\right) \\
& -\frac{\Delta t}{\Delta x}\left[\phi\left(r^+_{j+\frac{1}{2}}\right)\alpha^+_{j+\frac{1}{2}}\left(\Delta\hat{f}_{j+\frac{1}{2}}\right)^+ - \phi\left(r^-_{j+\frac{1}{2}}\right)\alpha^-_{j+\frac{1}{2}}\left(\Delta\hat{f}_{j+\frac{1}{2}}\right)^- - \right. \\
& \left. \phi\left(r^+_{j-\frac{1}{2}}\right)\alpha^+_{j-\frac{1}{2}}\left(\Delta\hat{f}_{j-\frac{1}{2}}\right)^+ + \phi\left(r^-_{j-\frac{1}{2}}\right)\alpha^-_{j-\frac{1}{2}}\left(\Delta\hat{f}_{j-\frac{1}{2}}\right)^-\right]
\end{aligned}
$$

$F^E_{j+\frac{1}{2}}$ is a numerical entropy satisfying flux

$$\left(\Delta\hat{f}_{j+\frac{1}{2}}\right)^+ = -\left(F^E_{j+\frac{1}{2}} - \hat{f}_{j+1}\right)$$

$$\left(\Delta\hat{f}_{j+\frac{1}{2}}\right)^- = \left(F^E_{j+\frac{1}{2}} - \hat{f}_{j}\right)$$

$$r^\pm_{j+\frac{1}{2}} = \left[\frac{\alpha^\pm_{j-\frac{1}{2}}\left(\Delta\hat{f}_{j-\frac{1}{2}}\right)^\pm}{\alpha^\pm_{j+\frac{1}{2}}\left(\Delta\hat{f}_{j+\frac{1}{2}}\right)^\pm}\right]^{\pm 1}$$

$$\alpha^\pm_{j+\frac{1}{2}} = \frac{1}{2}\left(1 \mp C^\pm_{j+\frac{1}{2}}\right)$$

$$C^\pm_{j+\frac{1}{2}} = \frac{\Delta t}{\Delta x}\frac{\left(\Delta f_{j+\frac{1}{2}}\right)^\pm}{\Delta u_{j+\frac{1}{2}}}$$

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[LV] R.J. LeVeque, Numerical Methods for Conservation Laws, Lecture Notes in Mathematics, ETH Zürich, Birkhäuser, 1992.