16.901: Sample Homework # 2
Solution

In this homework, we will consider numerical solutions to the one-dimensional diffusion equation,

$$\frac{\partial U}{\partial t} = \nu \frac{\partial^2 U}{\partial x^2},$$

where ν is a positive constant. Specifically, consider a forward Euler time integration and a 2nd-order centered-difference approximation in space,

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \nu \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2}.$$

1. Perform a semi-discrete Fourier analysis of this discretization and determine the eigenvalues $\lambda_m(\beta_m)$.

   Solution: Substituting $U^j = \hat{U}_m \exp(ik_m j\Delta x)$ into the semi-discrete equation (i.e. not discretizing in time) gives:

$$\begin{aligned}
\frac{dU_j}{dt} &= \nu \frac{U_{j+1} - 2U_j + U_{j-1}}{\Delta x^2}, \\
\frac{d\hat{U}_m}{dt}\exp(ij\beta_m) &= \nu \frac{\exp(i\beta_m) - 2 + \exp(-i\beta_m)}{\Delta x^2}\hat{U}_m \exp(ij\beta_m), \\
\frac{d\hat{U}_m}{dt} &= \nu \frac{\exp(i\beta_m) - 2 + \exp(-i\beta_m)}{\Delta x^2}\hat{U}_m, \\
\lambda_m \Delta t &= \frac{\nu\Delta t}{\Delta x^2}\left[\exp(i\beta_m) - 2 + \exp(-i\beta_m)\right], \\
\lambda_m \Delta t &= \frac{\nu\Delta t}{\Delta x^2}\left[\cos\beta_m + i\sin\beta_m - 2 + \cos\beta_m - i\sin\beta_m\right], \\
\Rightarrow \lambda_m \Delta t &= -2\frac{\nu\Delta t}{\Delta x^2}\left[1 - \cos\beta_m\right].
\end{aligned}$$

   Note that these eigenvalues are purely real and negative, as one would expect since this problem is pure diffusion.

2. What is the largest value of $\nu\Delta t/\Delta x^2$ for which the discretization is stable when integrated with forward Euler?

   Solution: Since $1 - \cos\beta_m \geq 0$, then the largest magnitude eigenvalue will occur when this term is largest. This occurs when $\beta_m = \pm\pi$, for which $1 - \cos\beta_m = 2$. Thus, the maximum magnitude of $\lambda_m \Delta t$ is,

$$\lambda_m(\pi)\Delta t = -4\frac{\nu\Delta t}{\Delta x^2}.$$

   For forward Euler, the negative real extent of the stability region is at $|\lambda\Delta t| = 2$. Thus, the largest magnitude (negative) eigenvalue must be inside this limit to remain stable:

$$\begin{aligned}
4\frac{\nu\Delta t}{\Delta x^2} &\leq 2 \\
\Rightarrow \frac{\nu\Delta t}{\Delta x^2} &\leq \frac{1}{2}
\end{aligned}$$