16.901: Sample Homework # 1
Solution

1. Consider the convection-diffusion equation,

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \nu\frac{\partial^2 T}{\partial x^2},$$

where ν is a constant. Assuming a wave of the form, $T(x,t) = \hat{T}_k(t)\exp(ikx)$, determine the solution for $\hat{T}_k(t)$ given the initial condition $\hat{T}_k(0) = 1$. Using this solution, show that the amplitude of $\hat{T}_k(t)$ decays as $t \to \infty$ only if $\nu > 0$.

Solution: Substituting $T(x,t) = \hat{T}_k(t)\exp(ikx)$ into the governing equation gives:

$$\begin{aligned}
\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} &= \nu\frac{\partial^2 T}{\partial x^2}, \\
\frac{d\hat{T}_k}{dt}\exp(ikx) + iku\hat{T}_k\exp(ikx) &= (ik)^2\nu\hat{T}_k\exp(ikx) \\
\frac{d\hat{T}_k}{dt} &= \left(-\nu k^2 - iku\right)\hat{T}_k. \\
\Rightarrow \hat{T}_k(t) &= \exp\left[\left(-\nu k^2 - iku\right)t\right].
\end{aligned}$$

Note, the solution has been chosen such that $\hat{T}_k(0) = 1$. Since the amplitude is unchanged by the imaginary portion of time dependence,

$$\begin{aligned}
\left|\hat{T}_k(t)\right| &= \left|\exp\left[\left(-\nu k^2 + iku\right)t\right]\right|, \\
&= \left|\exp\left(-\nu k^2 t\right)\right|\left|\exp\left(-ikut\right)\right|, \\
&= \left|\exp\left(-\nu k^2 t\right)\right|.
\end{aligned}$$

Clearly, this decreases as $t \to \infty$ since νk^2 is a positive number.

2. Consider the fourth-order equation:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \sigma\frac{\partial^4 T}{\partial x^4},$$

where σ is a constant. Assuming a wave of the form, $T(x,t) = \hat{T}_k(t)\exp(ikx)$, determine the solution for $\hat{T}_k(t)$ given the initial condition $\hat{T}_k(0) = 1$. Using this solution, for what values of σ will the amplitude of $\hat{T}_k(t)$ decay as $t \to \infty$?

Solution: Following the same procedure as in the previous problem, substituting $T(x,t) = \hat{T}_k(t)\exp(ikx)$ into the governing equation gives:

$$\begin{aligned}
\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} &= \sigma\frac{\partial^4 T}{\partial x^4}, \\
\frac{d\hat{T}_k}{dt}\exp(ikx) + iku\hat{T}_k\exp(ikx) &= (ik)^4\sigma\hat{T}_k\exp(ikx) \\
\frac{d\hat{T}_k}{dt} &= \left(\sigma k^4 - iku\right)\hat{T}_k. \\
\Rightarrow \hat{T}_k(t) &= \exp\left[\left(\sigma k^4 - iku\right)t\right].
\end{aligned}$$

Note, the solution has been chosen such that $\hat{T}_k(0) = 1$. Since the amplitude is unchanged by the imaginary portion of time dependence,

$$\begin{aligned}
\left|\hat{T}_k(t)\right| &= \left|\exp\left[\left(\sigma k^4 - iku\right)t\right]\right|, \\
&= \left|\exp\left(\sigma k^4 t\right)\right|\left|\exp\left(-ikut\right)\right|, \\
&= \left|\exp\left(\sigma k^4 t\right)\right|.
\end{aligned}$$

Clearly, for this to decrease as $t \to \infty$, σ must be negative: $\sigma < 0$.