Lecture 13

The Finite Element Method for Two-Dimensional Diffusion

In this lecture, we will consider the finite element approximation of the two-dimensional diffusion problem,

$$\nabla \cdot (k \nabla T) + q = 0. \tag{13.1}$$

As in the previous discussion of the method of weighted residuals and the finite element method, the approximate solution will have the form,

$$\tilde{T}(x, y) = \sum_{i=1}^{N} a_i \phi_i(x, y),$$

where $\phi_i(x, y)$ are the known basis functions and the a_i are the unknown weights to be determine for the specific problem. Following the Galerkin method of weighted residuals, we will weight Equation (13.1) by one of the basis functions and integrate the diffusion term by parts to give the following weighted residual,

$$R_j \equiv \int_{\delta\Omega} \phi_j \, k \nabla \tilde{T} \cdot \vec{n} \, ds - \int_{\Omega} \nabla \phi_j \cdot \left(k \nabla \tilde{T} \right) \, dA + \int_{\Omega} \phi_j q \, dA = 0. \tag{13.2}$$

13.1 Reference Element and Linear Elements

In multiple dimensions, a common practice in defining the polynomial functions within an element is to transform each element into a canonical, or so-called 'reference' element. Figure 13.1 shows the mapping commonly used for triangular elements which maps a generic triangle in (x, y) into a right triangle in (ξ_1, ξ_2).

In the reference element space, the nodal basis for linear polynomials will be one at one of the nodes, and reduce linearly to zero at the other nodes. These functions are,

$$\begin{aligned} \phi_1(\xi_1, \xi_2) &= 1 - \xi_1 - \xi_2, && (13.3) \\ \phi_2(\xi_1, \xi_2) &= \xi_1, && (13.4) \\ \phi_3(\xi_1, \xi_2) &= \xi_2. && (13.5) \end{aligned}$$

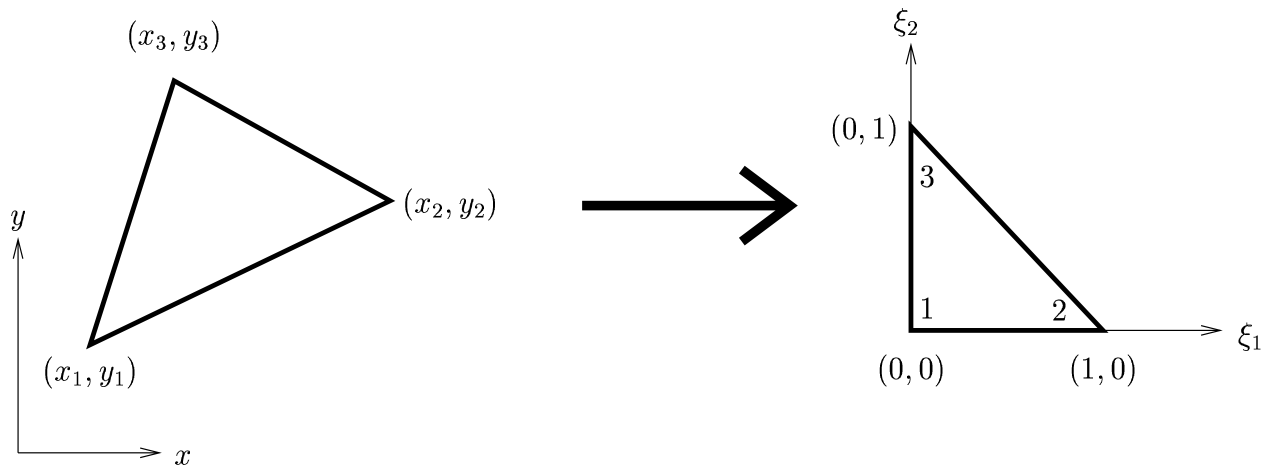

Figure 13.1: Transformation of a generic triangular element in (x, y) into the reference element in (ξ_1, ξ_2).

Then, within the element, the solution $\tilde{T}$ in the (ξ_1, ξ_2) space is the combination of these three basis functions multiplied by the corresponding nodal weights,

$$\tilde{T}(\xi_1, \xi_2) = \sum_{i=1}^{3} a_i \phi_i(\xi_1, \xi_2). \tag{13.6}$$

Using Equation (13.6), the value of $\tilde{T}$ can be found at any (ξ_1, ξ_2). To find the (x, y) locations in terms of the (ξ_1, ξ_2), we can expand them using the nodal locations and the nodal basis functions, i.e.,

$$\vec{x}(\xi_1, \xi_2) = \sum_{i=1}^{3} \vec{x}_i \phi_i(\xi_1, \xi_2).$$

Since the $\phi_i(\xi_1, \xi_2)$ are linear functions of ξ_1 and ξ_2, this amounts to a linear transformation between (ξ_1, ξ_2) and (x, y). Specifically, substituting the nodal basis functions gives,

$$\begin{aligned} \vec{x}(\xi_1, \xi_2) &= \vec{x}_1 (1 - \xi_1 - \xi_2) + \vec{x}_2 \xi_1 + \vec{x}_3 \xi_2, \\ \Rightarrow \vec{x}(\xi_1, \xi_2) &= \vec{x}_1 + (\vec{x}_2 - \vec{x}_1)\, \xi_1 + (\vec{x}_3 - \vec{x}_1)\, \xi_2. \end{aligned}$$

This can be written in a matrix notation as,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{13.7}$$

This equation can be inverted to also determine (ξ_1, ξ_2) as a function of (x, y),

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}^{-1} \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix} \tag{13.8}$$

13.2 Differentiation using the Reference Element

To find the derivative of $\tilde{T}$ with respect to x (or similarly y) within an element, we differentiate the three nodal basis functions within the element,

$$\begin{aligned} \tilde{T}_x &= \frac{\partial}{\partial x}\left(\sum_{i=1}^{3} a_i \phi_i\right), \\ &= \sum_{i=1}^{3} a_i \frac{\partial \phi_i}{\partial x}. \end{aligned}$$

To find the x-derivatives of each of the ϕ_i's, the chain rule is applied,

$$\frac{\partial \phi_i}{\partial x} = \frac{\partial \phi_i}{\partial \xi_1}\frac{\partial \xi_1}{\partial x} + \frac{\partial \phi_i}{\partial \xi_2}\frac{\partial \xi_2}{\partial x}.$$

Similarly, to find the y-derivatives, the derivatives with respect to y,

$$\frac{\partial \phi_i}{\partial y} = \frac{\partial \phi_i}{\partial \xi_1}\frac{\partial \xi_1}{\partial y} + \frac{\partial \phi_i}{\partial \xi_2}\frac{\partial \xi_2}{\partial y}.$$

The calculation of the derivatives of ϕ_i with respect to the ξ's gives,

$$\begin{aligned} \frac{\partial \phi_1}{\partial \xi_1} &= -1, \qquad \frac{\partial \phi_1}{\partial \xi_2} = -1, \\ \frac{\partial \phi_2}{\partial \xi_1} &= 1, \qquad \frac{\partial \phi_2}{\partial \xi_2} = 0, \\ \frac{\partial \phi_3}{\partial \xi_1} &= 0, \qquad \frac{\partial \phi_3}{\partial \xi_2} = 1. \end{aligned}$$

The only remaining terms are the calculation of $\frac{\partial \xi_1}{\partial x}$, $\frac{\partial \xi_2}{\partial x}$, etc. which can be found by differentiating Equation (13.8),

$$\begin{aligned} \frac{\partial \vec{\xi}}{\partial \vec{x}} &= \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}^{-1}, \\ &= \frac{1}{J}\begin{pmatrix} y_3 - y_1 & -(x_3 - x_1) \\ -(y_2 - y_1) & x_2 - x_1 \end{pmatrix}, \end{aligned}$$

where

$$J = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1).$$

Note that the Jacobian, J, is equal to twice the area of the triangular element.

13.3 Construction of the Stiffness Matrix

The stiffness matrix arises in the calculation of $\int_\Omega \nabla \phi_j \cdot \left(k \nabla \tilde{T}\right)\, dA$. As in the one-dimensional case, the j-th row of the stiffness matrix K corresponds to the weighted residual of ϕ_j. The

i-th column in the j-th row corresponds to the dependence of the j-th weighted residual on a_i. Further drawing on the one-dimensional example, the weighted residuals are assembled by calculating the contribution to all of the residuals from within a single element. In the two-dimensioal linear element situation, three weighted residuals are impacted by a given element, specifically, the weighted residuals corresponding to the nodal basis functions of the three nodes of the triangle. For example, in each element we must calculate,

$$\int_{\Omega_e} \nabla\phi_1 \cdot \left(k\nabla\tilde{T}\right)\, dA, \qquad \int_{\Omega_e} \nabla\phi_2 \cdot \left(k\nabla\tilde{T}\right)\, dA, \qquad \int_{\Omega_e} \nabla\phi_3 \cdot \left(k\nabla\tilde{T}\right)\, dA,$$

where Ω_e is spatial domain for a specific element. As described in Section 13.2, the gradient of $\tilde{T}$ can be written,

$$\nabla\tilde{T}(x,y) = \sum_{i=1}^{3} a_i \nabla\phi_i(x,y),$$

thus the weighted residuals expand to,

$$\int_{\Omega} \nabla\phi_j \cdot \left(k\nabla\tilde{T}\right)\, dA = \sum_{i=1}^{3} a_i K_{j,i}, \qquad \text{where} \qquad K_{j,i} \equiv \int_{\Omega} \nabla\phi_j \cdot (k\nabla\phi_i)\, dA.$$

For the situation in which k is constant, and linear elements are used, then this reduces to,

$$K_{j,i} \equiv k\nabla\phi_j \cdot \nabla\phi_i A_e$$

where A_e is the area of element e.

13.4 Integration in the Reference Element

The reference element can also be used to evaluate integrals. For example, consider the evaluation of the forcing function integral within an element,

$$\int_{\delta\Omega_k} w(\vec{x})\, q(\vec{x})\, dA.$$

In transforming the integral from (x,y) to (ξ_1,ξ_2), the differential area of integration must be transform using the following result,

$$dA = dx\, dy = J d\xi_1\, d\xi_2 = J\, dA_\xi. \tag{13.9}$$

Thus, the integrals can now be evaluated in reference element space,

$$\int_{\Omega_\xi} w(\vec{x}(\vec{\xi}))\, q(\vec{x}(\vec{\xi}))\, J dA_\xi.$$

In-class Discussion 13.1 (Calculation of the Mass Matrix)