Lecture 11

Method of Weighted Residuals

In this lecture, we introduce the method of weighted residuals which provides the most general formulation for the finite element method. To begin, let's focus on the particular problem of steady heat diffusion in a rod. This problem can be modeled as a one-dimensional PDE for the temperature, T:

$$(kT_x)_x = -q, \tag{11.1}$$

where $k(x)$ thermal conductivity of the material and $q(x)$ is the heat source (per unit area), respectively. Note that both k and q could be functions of x. Also, let the physical domain for the problem be from $x = -L/2$ to $x = L/2$.

Example 11.1 (Steady heat diffusion) *Suppose that the rod has a length of $L = 2$, the thermal conductivity is constant, $k = 1$, and the heat source, $q(x) = 50e^x$. Assume that the temperature at the ends of the rod are to be maintained at $T(\pm 1) = 100$. Equation (11.1) can be integrated twice to obtain:*

$$\begin{aligned}
(kT_x)_x &= -q, \\
T_{xx} &= -50e^x, \\
T_x &= -50e^x + a, \\
T &= -50e^x + ax + b.
\end{aligned}$$

Now, applying boundary conditions so that $T(\pm 1) = 100$,

$$\begin{aligned}
-50e^1 + a + b &= 100, \\
-50e^{-1} - a + b &= 100.
\end{aligned}$$

This is a 2×2 system which can be solved for a and b,

$$\begin{aligned}
a &= 50 \sinh 1 \\
b &= 100 + 50 \cosh 1,
\end{aligned}$$

where $\cosh y = (e^y + e^{-y})/2$ and $\sinh y = (e^y - e^{-y})/2$. Thus, the exact solution is,

$$T = -50e^x + 50x \sinh 1 + 100 + 50 \cosh 1. \tag{11.2}$$

A plot of this solution is shown in Figure 11.1.

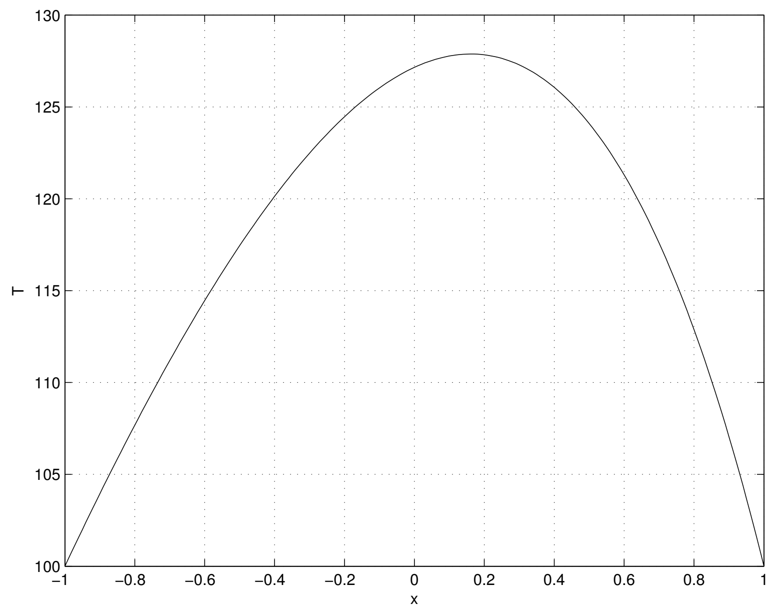


Figure 11.1: Temperature distribution for $q = 50e^x$, $L = 2$, and $k = 1$.

A common approach to approximating the solution to a PDE such as heat diffusion is to use a series of weighted functions. For example, for the temperature in Example 11.1 we might assume that,

$$\tilde{T}(x) = 100 + \sum_{i=1}^{N} a_i \phi_i(x),$$

where N is the number of terms (functions) in the approximation, $\phi_i(x)$ are the (known) functions, and a_i are the unknown function weights. The functions $\phi_i(x)$ are usually designed to satisfy the boundary conditions. So, in this example where the temperature is 100 at $x = \pm 1$, then $\phi_i(\pm 1) = 0$ (don't forget that $\tilde{T}$ was defined to include the constant term of 100).

The question remains what functions (and how many) to choose for $\phi_i(x)$. While many good choices exist, we will use polynomials in x because polynomial approximations are used extensively in finite element methods (our main interest). For this problem, the following ideas can be used to determine the form of the $\phi_i(x)$:

- First, we note that requiring $\phi_i(\pm 1) = 0$ places two conditions on each $\phi_i(x)$. These two conditions can be satisfied with a linear function of x, but the linear function which equals 0 at $x = \pm 1$ is simply 0. Since this does not add anything to the solution even after multiplying by a weight, the first non-trivial function would be a quadratic function.

- A quadratic function can be designed to satisy the boundary conditions in the following manner,

$$\phi_1(x) = (1 + x)(1 - x).$$

By including factors which go to zero at the end points, we have constructed a quadratic function which will satisfy the required boundary conditions. A plot of $\phi_1(x)$ is shown in Figure 11.2.

- Suppose we wanted to include a cubic polynomial in the approximation, then one way we could do this is multiply $\phi_1(x)$ by x.

$$\phi_2(x) = x\phi_1(x) = x(1+x)(1-x).$$

Since $\phi_1(x)$ goes to zero at the end points, then so will $\phi_2(x)$. A plot of $\phi_2(x)$ is shown in Figure 11.2. There are actually some better ways to choose these higher-order polynomials then simply multiplying the lowest order polynomial by powers of x. The problem with the current approach is that if the number of terms were large (so that the powers of x would be large), then the set of polynomials (i.e. $\phi_i(x)$) become very poorly conditioned resulting in many numerical difficulties. We will not discuss issues of conditioning but more advanced texts on finite element methods or related subjects can be consulted. For low order polynomial approximations, the issues of conditioning do not play an important role.

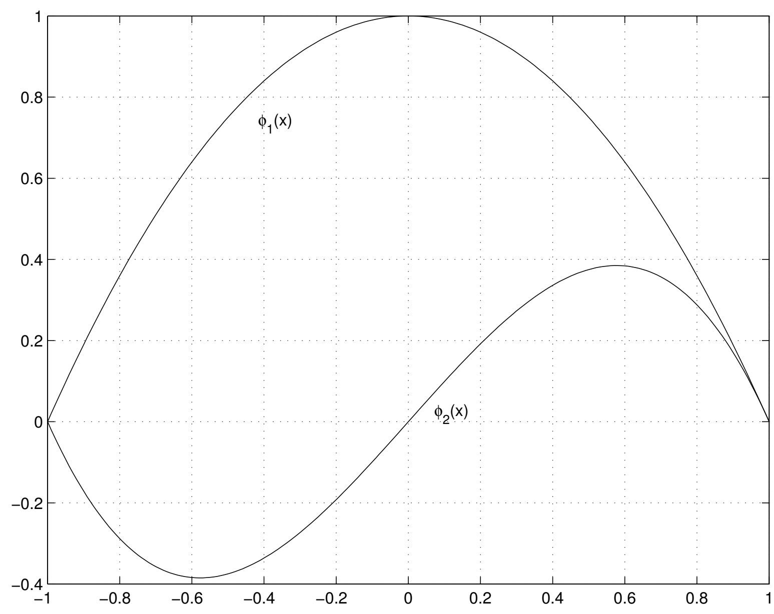


Figure 11.2: $\phi_1(x)$ and $\phi_2(x)$ for example heat diffusion problem.

Having selected a set of functions, we must now develop a way to determine values of a_i that will lead to a good approximation of the actual $T(x)$. While several ways exist to do this, we will focus on two methods in these introductory notes: the collocation method and the method of weighted residuals.

11.1 The Collocation Method

One approach to determine the N values of a_i would be to enforce the governing PDE at N points. Note that in general, the exact solution will not be a linear combination of the $\phi_i(x)$, so it will not be possible to enforce the PDE at every point in the domains. To see this, let's substitute $\tilde{T}(x)$ into Equation (11.1). Note that,

$$\begin{aligned}
\tilde{T}_{xx} &= \frac{\partial^2}{\partial x^2}\left[a_1\phi_1(x) + a_2\phi_2(x)\right], \\
(\phi_1)_{xx} &= -2, \\
(\phi_2)_{xx} &= -6x, \\
\Rightarrow \tilde{T}_{xx} &= -2a_1 - 6a_2 x.
\end{aligned}$$

Next, we define a residual for Equation (11.1),

$$R(\tilde{T}, x) \equiv \left(k\tilde{T}_x\right)_x + q.$$

If the solution were exact, then the $R = 0$ for all x. Now, substitution of our chosen $\tilde{T}$ into the residual (recall $k = 1$ and $q = 50e^x$ in this example) gives,

$$R(\tilde{T}, x) = -2a_1 - 6a_2 x + 50e^x. \tag{11.3}$$

Clearly, since a_1 and a_2 are constants (i.e. they do not depend on x), there is no way for this residual to be zero for all x.

The question remains, where should the N points be selected. The points at which the governing equation will be enforced are known as the collocation points. We will choose the relatively simple approach of equally subdividing the domain with $N = 2$ interior collocation points. For this domain from $-1 \le x \le 1$, the equi-distant collocation points would be at $x = \pm 1/3$. Thus, the two conditions for determining a_1 and a_2 are,

$$\begin{aligned}
R(\tilde{T}, -1/3) &= 0, \\
R(\tilde{T}, 1/3) &= 0.
\end{aligned}$$

From Equation (11.3) this gives,

$$\begin{aligned}
-2a_1 + 2a_2 + 50e^{-1/3} &= 0, \\
-2a_1 - 2a_2 + 50e^{1/3} &= 0.
\end{aligned}$$

Re-arranging this into a matrix form gives,

$$\begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -50e^{-1/3} \\ -50e^{1/3} \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 25\cosh 1/3 \\ 25\sinh 1/3 \end{pmatrix} = \begin{pmatrix} 26.402 \\ 8.489 \end{pmatrix}.$$

The results using this collocation method are shown in Figure 11.3 which includes plots of $\tilde{T}$, the error (i.e. $\tilde{T} - T$), and the residual. Note that the residual is clearly exactly zero at the collocation points (i.e. $x = \pm 1/3$), though the approximation is not exact at these points (i.e. $\tilde{T} \neq T$ at $x = \pm 1/3$).

11.2 The Method of Weighted Residuals

While the collocation method enforces the residual to be zero at N points, the method of weighted residuals requires N weighted integrals of the residual to be zero. A weighted residual is simply the integral of a weight function, $w(x)$ and the residual over the domain. For example, in the one-dimensional diffusion problem we are considering, a weighted residual is,

$$\int_{-1}^{1} w(x)\, R(\tilde{T}, x)\, dx.$$

By choosing N weight functions, $w_j(x)$ for $1 \leq j \leq N$ and setting these N weighted residuals to zero, we may determine N values of a_i. We define the weighted residual for $w_j(x)$ to be,

$$R_j(\tilde{T}) \equiv \int_{-1}^{1} w_j(x)\, R(\tilde{T}, x)\, dx.$$

And, the method of weighted residuals requires,

$$R_j(\tilde{T}) = 0 \qquad \text{for } 1 \leq j \leq N.$$

In the method of weighted residuals, the next step is to determine appropriate weight functions. A common approach, known as the Galerkin method, is to set the weight functions equal to the functions used to approximate the solution. That is,

$$w_j(x) = \phi_j(x). \quad \text{(Galerkin)}.$$

For the heat diffusion example we have been considering,

$$\begin{aligned} w_1(x) &= (1-x)(1+x), \\ w_2(x) &= x(1-x)(1+x). \end{aligned}$$

Now, we must calculate the weighted residuals. For the example,

$$\begin{aligned} R_1(\tilde{T}) &= \int_{-1}^{1} w_1(x)\, R(\tilde{T}, x)\, dx, \\ &= \int_{-1}^{1} (1-x)(1+x)\, \left(-2a_1 - 6a_2 x + 50e^x\right)\, dx, \\ &= -\frac{8}{3}a_1 + 200e^{-1}. \end{aligned}$$

To do this integral, the following results were used (the constants of integration are neglected),

$$\begin{aligned} \int (1-x)(1+x)\, dx &= x - \frac{1}{3}x^3, \\ \int x(1-x)(1+x)\, dx &= \frac{1}{2}x^2 - \frac{1}{4}x^4, \\ \int x^2 e^x\, dx &= x^2 e^x - 2xe^x + 2e^x. \end{aligned}$$

Similarly, calculating R_2:

$$\begin{aligned}
R_2(\tilde{T}) &= \int_{-1}^{1} w_2(x)\, R(\tilde{T}, x)\, dx, \\
&= \int_{-1}^{1} x(1-x)(1+x)\, \left(-2a_1 - 6a_2 x + 50e^x\right)\, dx, \\
&= -\frac{8}{5}a_2 + 100e^1 - 1200e^{-1},
\end{aligned}$$

where the following results have been used,

$$\begin{aligned}
\int x^2(1-x)(1+x)\, dx &= \frac{1}{3}x^3 - \frac{1}{5}x^5, \\
\int xe^x\, dx &= xe^x - e^x, \\
\int x^3e^x\, dx &= x^3e^x - 3x^2e^x + 6xe^x - 6e^x.
\end{aligned}$$

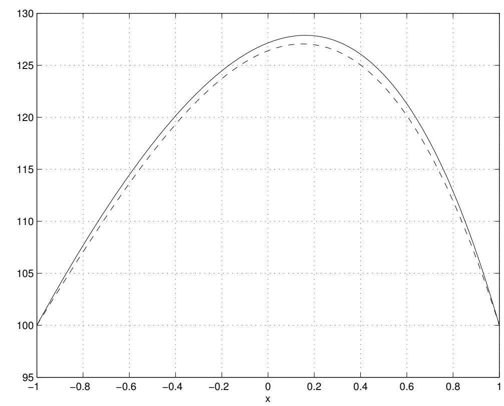
Finally, we can solve for a_1 and a_2 by setting the weighted residuals R_1 and R_2 to zero,

$$\begin{aligned}
-\frac{8}{3}a_1 + 200e^{-1} &= 0, \\
-\frac{8}{5}a_2 + 100e^1 - 700e^{-1} &= 0.
\end{aligned}$$

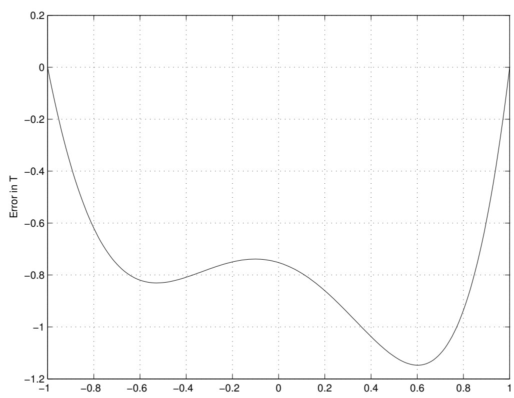
This could be written as a 2×2 matrix equation and solved, but the equations are decoupled and can be readily solved,

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 75e^{-1} \\ \frac{125}{2}e^1 - \frac{875}{2}e^{-1} \end{pmatrix} = \begin{pmatrix} 27.591 \\ 8.945 \end{pmatrix}.$$

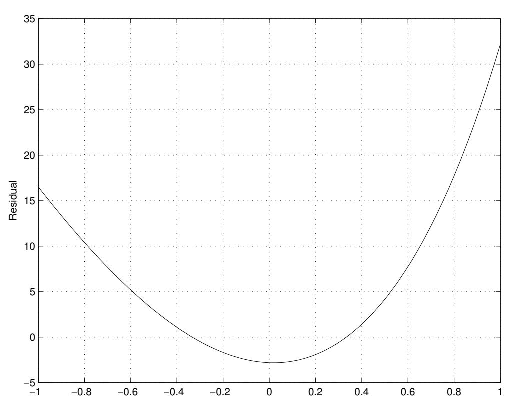
The results using this method of weighted residuals are shown in Figure 11.4. Comparison with the collocation method results shows that the method of weighted residuals is clearly more accurate.



(a) Comparison of T (solid) and $\tilde{T}$ (dashed)

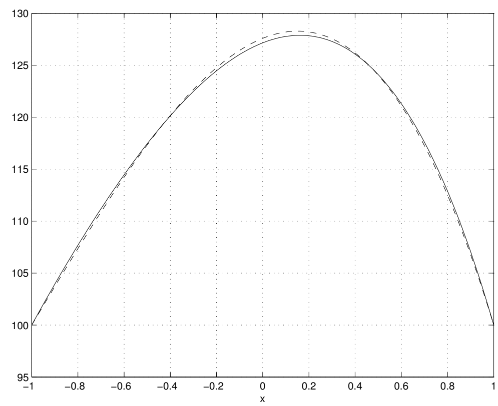


(b) Error, $\tilde{T} - T$

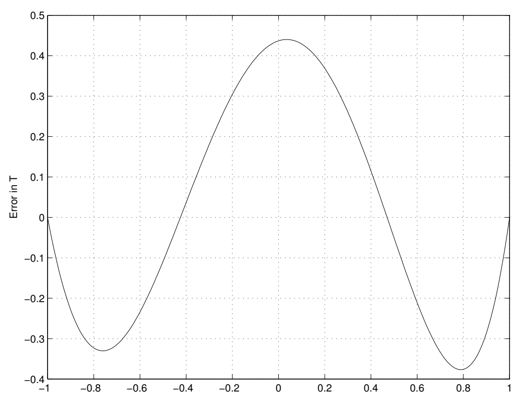


(c) Residual, $R(\tilde{T}, x)$

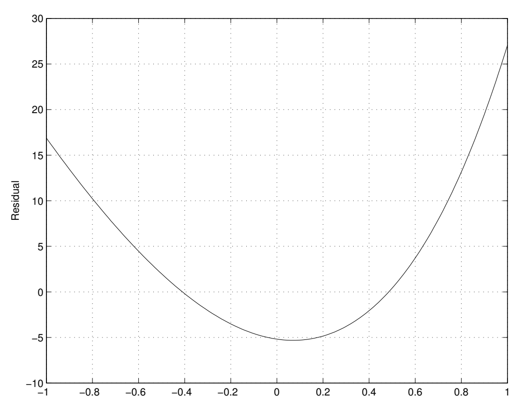
Figure 11.3: Results for collocation method.



(a) Comparison of T (solid) and $\tilde{T}$ (dashed)



(b) Error, $\tilde{T} - T$



(c) Residual, $R(\tilde{T}, x)$

Figure 11.4: Results for method of weight residuals.