16.901: Homework # 8
Solution

In this homework, you will investigate the convergence of the finite volume method applied to two-dimensional convection. Specifically, you will modify the Matlab script, **convect2d.m**

The problem to be studied in this homework will be for the square domain $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$. The velocity field will be constant and given by,

$$
\begin{aligned}
u &= 1 \\
v &= -1
\end{aligned}
$$

With this velocity field, the $x = -2$ boundary and the $y = 2$ boundary will be inflow boundaries. On these boundaries, the boundary conditions are,

$$ U(x,y) = e^{-10(y-1)^2}. $$

Modify the **convect2d.m** script to solve this problem. Perform simulations for grids with Nx = Ny = 20, 40, and 80, and 160. Be sure to run the simulations for a long enough time so that the steady state solution is reached. Use a CFL = 1.

For the four grids, plot the steady state solution on the bottom boundary (i.e. $y = -2$). Include the exact solution on the plot. Based on these results, what is your best guess for the order of accuracy of the method with respect to Δx?

Solution: The solution plots at the bottom boundaries are shown in Figure 1 for Nx = Ny = 20, 40, and 80, and 160. While the solutions are clearly improving, they still are far from the exact solution.

The errors for each grid are tabulated in Table 1. The order of accuracy between the different grids are estimated as follows. First, define the order of accuracy, p, as the order with which the error scales with the cell size Δx (in this problem $\Delta x = \Delta y$),

$$ E = O(\Delta x^p). $$

Then, the ratio of errors between grids of different Δx is,

$$
\begin{aligned}
\frac{E_2}{E_1} &= \left(\frac{\Delta x_2}{\Delta x_1}\right)^p, \\
\log E_2 - \log E_1 &= p\left(\log \Delta x_2 - \log \Delta x_1\right), \\
\Rightarrow p &= \frac{\log E_2 - \log E_1}{\log \Delta x_2 - \log \Delta x_1}.
\end{aligned}
$$

As can be seen in Table 1, the convergence rate is still increasing with the mesh resolution. Note: I ran additional meshes in an attempt to reach the asymptotic convergence rate but clearly p is still increasing. A log-log plot of the error versus mesh size is also shown in Figure 2 which shows that the order of accuracy (which is the slope in this log-log plot) is still increasing at the finest resolution.
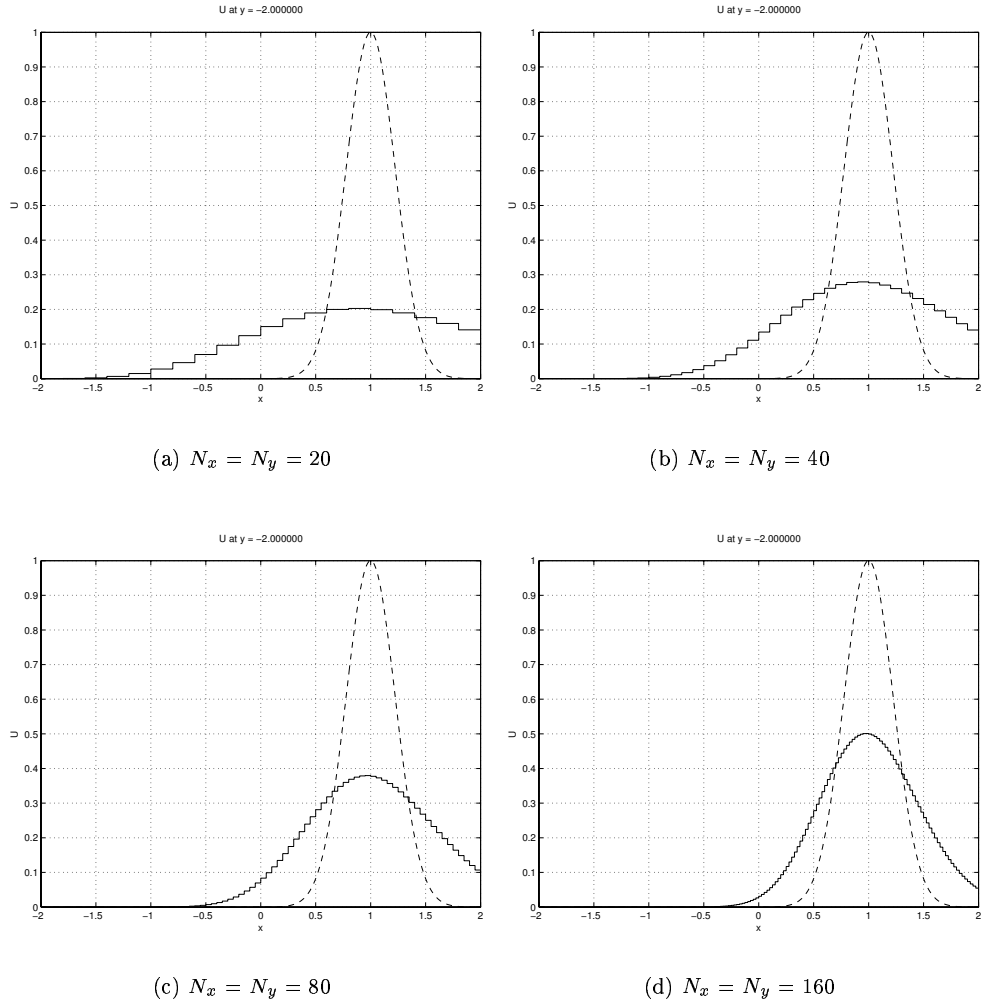
(a) $N_x = N_y = 20$

(b) $N_x = N_y = 40$

(c) $N_x = N_y = 80$

(d) $N_x = N_y = 160$

Figure 1: Comparison of finite volume and exact solution along $y = -2$ for 2-d convection problem versus grid size.

Nx, Ny	$U(1, -2)$	Error	p
20	0.2028	0.7972	0.1457
40	0.2794	0.7206	0.2165
80	0.3794	0.6202	0.3117
160	0.5003	0.4997	0.4453
320	0.6330	0.3670	0.5895
640	0.7561	0.2439	

Table 1: Error in value at $(x, y) = (1, -2)$ for 2-d convection problem versus grid size (note the exact value is 1 at this location). Order of accuracy estimate is between the given mesh size and the next finest.
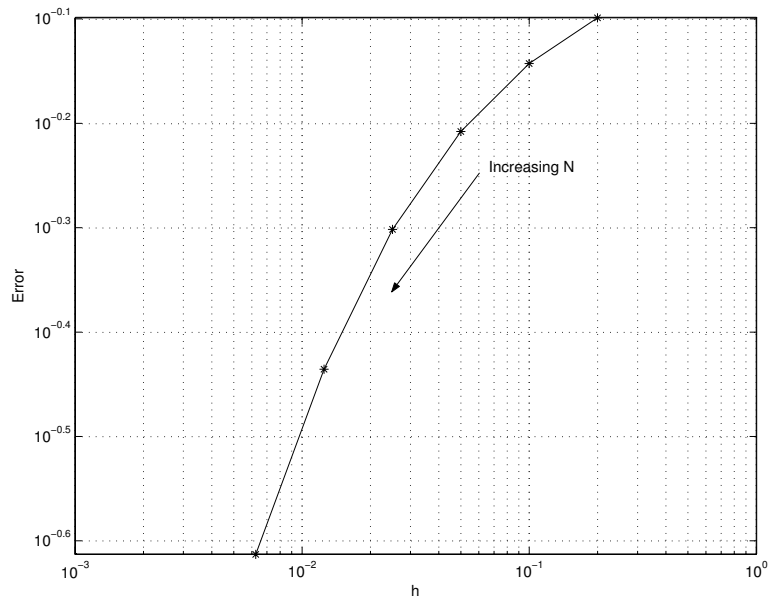
Figure 2: Error in value at $(x, y) = (1, -2)$ for 2-d convection problem versus h (grid spacing).