16.901: Homework # 5
Solution

Starting from the Matlab script, **dif1d.m** from Lecture 5 on ODE stiffness, implement the second-order backwards differentiation algorithm to solve the one-dimensional heat diffusion problem.

When running the simulation, set the number of points to Nx=1001. Run the simulation from $t = 0$ to $t = 0.5$ for three different values of $\Delta t = 0.1$, 0.01, and 0.001.

1. Store the value of the temperature at the middle of the domain (i.e. T at node 501) and plot it versus t. Overlay the results from all three Δt ON THE SAME GRAPH. Also, plot the distribution of $T(x)$ at the final time for all three values of Δt (again, overlay these on the same graph). To clarify: for this part of the problem, you should have only two graphs!

   Solution: See Figures 1 and 2.

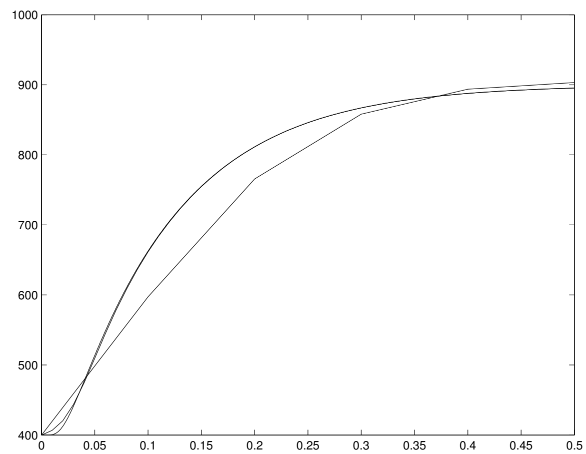

Figure 1: Behavior of T in middle of domain as a function of time for Δt =0.1, 0.01, and 0.001 for second-order backwards differentiation method.

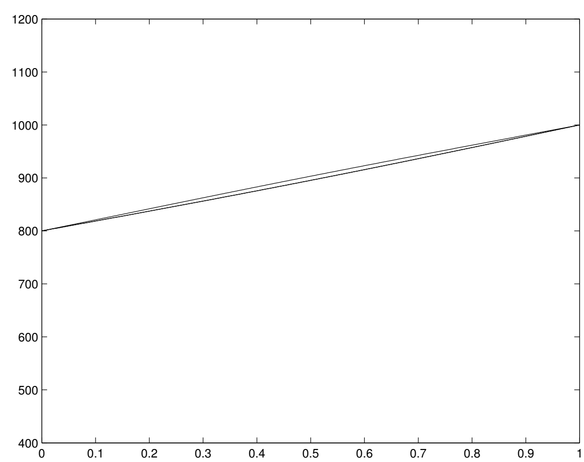

Figure 2: Distribution of T at $t = 0.5$ for Δt =0.1, 0.01, and 0.001 for second-order backwards differentiation method.

2. Perform the same three Δt simulations using the backward Euler method and make the same two plots as above.

   Solution: See Figures 3 and 4.

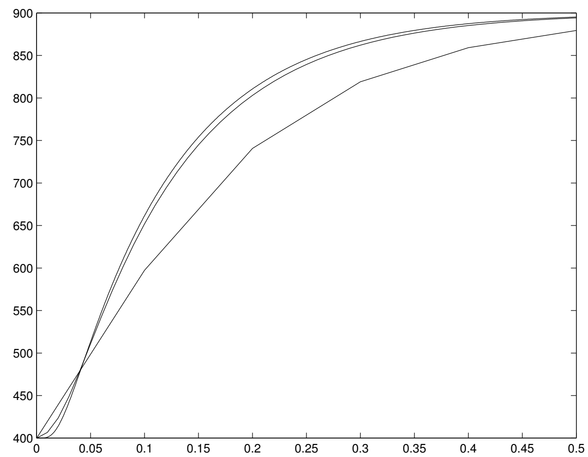

Figure 3: Behavior of T in middle of domain as a function of time for Δt =0.1, 0.01, and 0.001 for backwards Euler method.

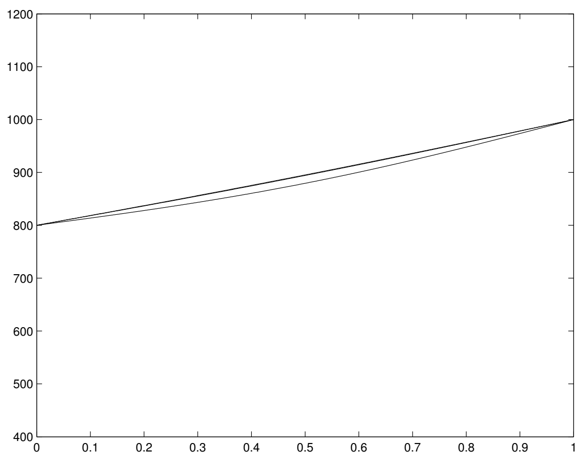

Figure 4: Distribution of T at $t = 0.5$ for Δt =0.1, 0.01, and 0.001 for backwards Euler method.

3. Perform the same three Δt simulations using the trapezoidal integration method and make the same two plots as above.

   Solution: See Figures 5 and 6.

4. Briefly compare the results for the three methods and three Δt. Are any of the methods clearly better for solving this problem.

   Solution: The second-order backwards differentiation has the most similar behavior for all Δt and would appear to be the best method. The backwards Euler method is also well-behaved but the lower accuracy (backward Euler is only first-order accurate) is apparent in the larger error for the largesr Δt results. The trapezoidal results are much more oscillatory. As discussed in class, this is due to the oscillatory nature of the trapezoidal scheme for large $\lambda\Delta t$ values. Specifically, the trapezoidal

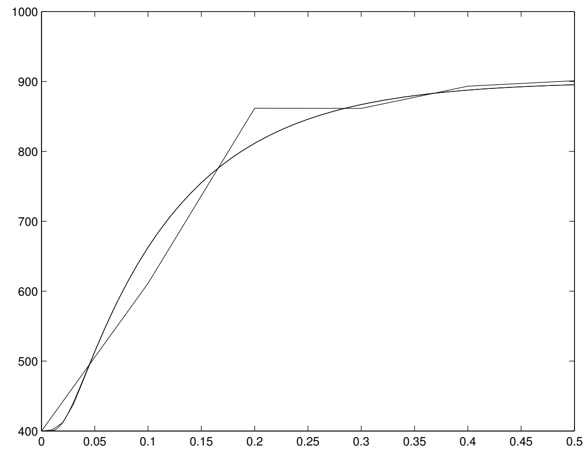

Figure 5: Behavior of T in middle of domain as a function of time for Δt =0.1, 0.01, and 0.001 for trapezoidal method.

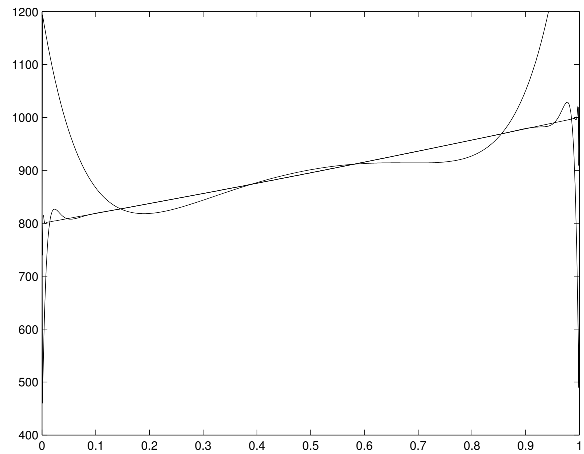

Figure 6: Distribution of T at $t = 0.5$ for Δt =0.1, 0.01, and 0.001 for trapezoidal method.

amplification factor is,

$$g = \frac{1 + \frac{1}{2}\lambda\Delta t}{1 - \frac{1}{2}\lambda\Delta t}$$

Thus when $\lambda\Delta t$ is large in magnitude, then $g \to -1$ from below. In the problem here (heat diffusion), some of the eigenvalues are in fact very large in magnitude, this we expect some oscillatory, slowly decaying error modes. This is evidently what we are observing in the trapezoidal solution results which shows clearly oscillatory behavior near the boundaries (at $x = 0$ and $x = 1$).