16.901: Homework # 3
Solution

1. The third-order Adams-Bashforth algorithm is,

$$v^{n+1} = v^n + \Delta t \left(\frac{23}{12} f^n - \frac{16}{12} f^{n-1} + \frac{5}{12} f^{n-2} \right).$$

Plot the eigenvalue stability region for this algorithm.

Solution: To determine the eigenvalue stability region, f is assumed to be a linear forcing, $f = \lambda u$,

$$v^{n+1} = v^n + \lambda \Delta t \left(\frac{23}{12} v^n - \frac{16}{12} v^{n-1} + \frac{5}{12} v^{n-2} \right).$$

Then, substituting the amplification factor, $v^n = g^n v^0$,

$$g^{n+1} = g^n + \lambda \Delta t \left(\frac{23}{12} g^n - \frac{16}{12} g^{n-1} + \frac{5}{12} g^{n-2} \right).$$

Re-arranging,

$$g^{n-2} \left[g^3 - g^2 - \lambda \Delta t \left(\frac{23}{12} g^2 - \frac{16}{12} g + \frac{5}{12} \right) \right] = 0.$$

Thus, the non-zero roots satisfy,

$$g^3 - g^2 - \lambda \Delta t \left(\frac{23}{12} g^2 - \frac{16}{12} g + \frac{5}{12} \right) = 0.$$

To find the stability boundary, which is where $|g| = 1$, substitute $g = e^{i\theta}$ and solve for $\lambda \Delta t$,

$$\lambda \Delta t = \frac{e^{3i\theta} - e^{2i\theta}}{\frac{23}{12} e^{2i\theta} - \frac{16}{12} e^{i\theta} + \frac{5}{12}}$$

A plot of this stability region is shown in Figure 1.

2. Consider the problem $u_t = -u^2$ with $u(0) = 1$. Estimate the maximum timestep for which this algorithm will be stable for this problem.

Solution: As discussed in lecture, we need to linearize the problem to estimate the eigenvalues. For this problem, $\lambda = \partial f / \partial u = -2u$. The maximum λ occurs when u is largest which happens at the initial condition. Thus, an estimate of the largest $\lambda = -2$ since $u(0) = 1$. Since this eigenvalue is a negative real number, the limiting timestep is set by the crossing of the stability boundary along the negative real axis. This crossing is at $\lambda = -0.545$. Thus, the maximum stable timestep is estimated to be $\Delta t = 0.27$.

3. Implement this algorithm to solve the above problem and through experimentation determine the maximum Δt for which the solution does not grow unbounded (make sure to run the algorithm to very large times T). Include plots of the solution for a stable value and an unstable value of Δt.

Solution: The third-order Adams-Bashforth algorithm was implemented in **ga_ab3.m**. Note: for the first step, the forward Euler method is used, and for the second step, the 2nd order Adams-Bashforth method is used. The algorithm was run until $T \approx 200$ for a variety of Δt. It was found that instability occurs when $\Delta t \geq 0.78$. Results for a stable and unstable Δt are shown in Figure 2.

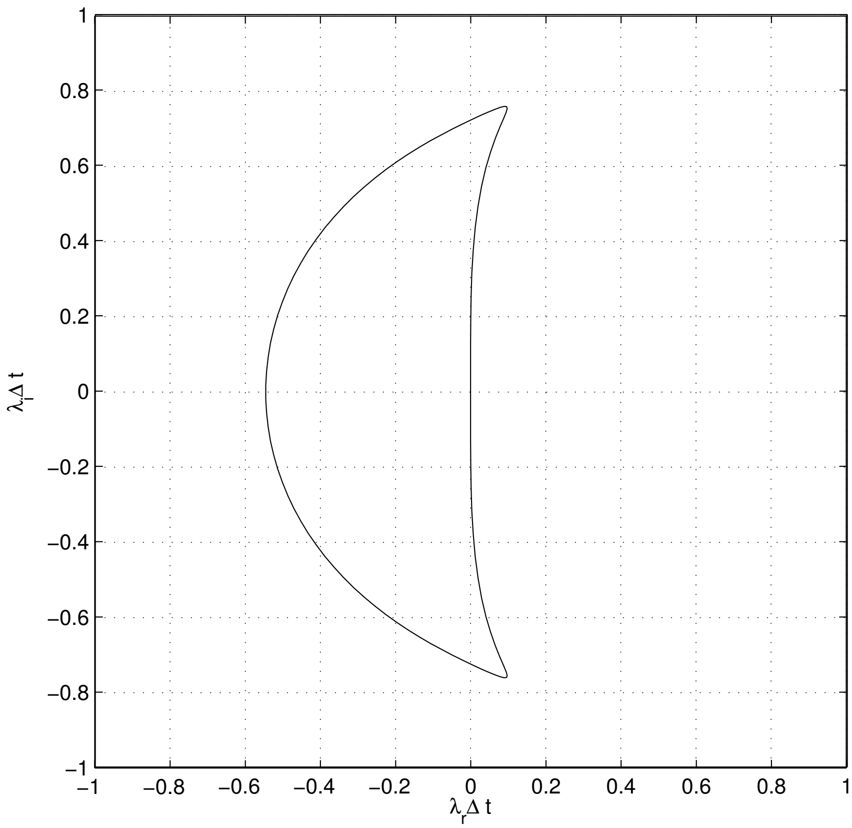

Figure 1: Stability of third-order Adams-Bashforth method.

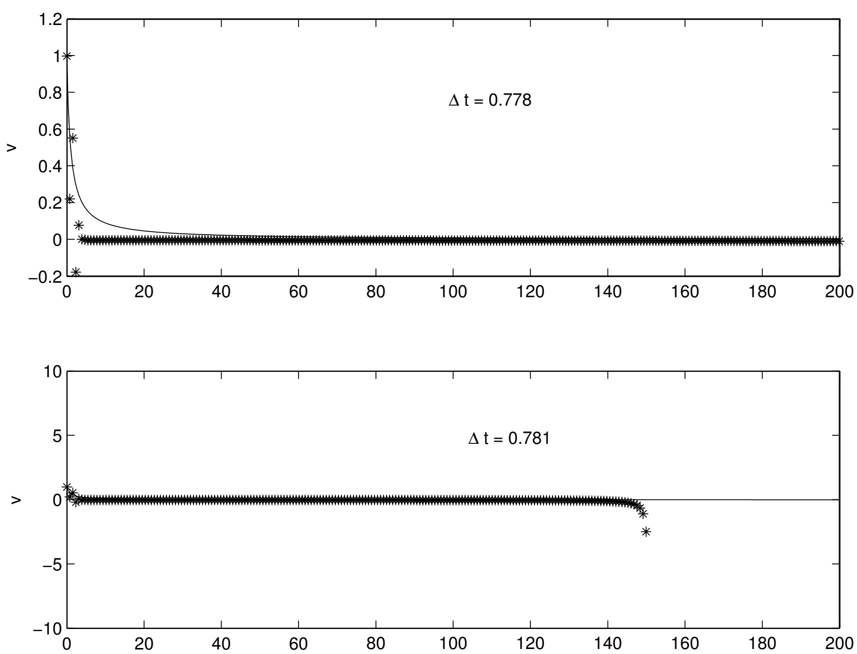

Figure 2: Third-order Adams-Bashforth approximation of $\dot{u} = -u^2$ with $u(0) = 1$ and $\Delta t = 0.778$ (stable) and 0.781 (unstable).