16.901: Homework # 1
Solution

Consider the following ODE integration method,

$$\frac{v^{n+1} - v^n}{\Delta t} = \frac{3}{2} f(v^n, t^n) - \frac{1}{2} f(v^{n-1}, t^{n-1}).$$

1. Determine the leading term of the local truncation error for this method. What is the local order of accuracy?

   Solution: Re-arranging the method,

   $$v^{n+1} = v^n + \Delta t \left[\frac{3}{2} f(v^n, t^n) - \frac{1}{2} f(v^{n-1}, t^{n-1}) \right] = N(v^n, v^{n-1}, \Delta t)$$

   Then the local truncation error is,

   $$\begin{aligned} \tau &\equiv N(u^n, u^{n-1}, \Delta t) - u^{n+1} \\ &= u^n + \Delta t \left[\frac{3}{2} f(u^n, t^n) - \frac{1}{2} f(u^{n-1}, t^{n-1}) \right] - u^{n+1} \\ &= u^n + \Delta t \left[\frac{3}{2} u_t^n - \frac{1}{2} u_t^{n-1} \right] - u^{n+1} \end{aligned}$$

   Next, substitute the following Taylor series,

   $$\begin{aligned} u^{n+1} &= u^n + \Delta t u_t^n + \frac{1}{2} \Delta t^2 u_{tt}^n + \frac{1}{6} \Delta t^3 u_{ttt}^n + O(\Delta t^4), \\ u_t^{n-1} &= u_t^n - \Delta t u_{tt}^n + \frac{1}{2} \Delta t^2 u_{ttt}^n + O(\Delta t^3), \end{aligned}$$

   which gives,

   $$\begin{aligned} \tau &= u^n + \Delta t \left\{ \frac{3}{2} u_t^n - \frac{1}{2} \left[u_t^n - \Delta t u_{tt}^n + \frac{1}{2} \Delta t^2 u_{ttt}^n + O(\Delta t^3) \right] \right\} \\ &\quad - \left[u^n + \Delta t u_t^n + \frac{1}{2} \Delta t^2 u_{tt}^n + \frac{1}{6} \Delta t^3 u_{ttt}^n + O(\Delta t^4) \right] \\ &= -\frac{5}{12} \Delta t^3 u_{ttt}^n + O(\Delta t^4) \end{aligned}$$

   Thus, the leading error term is $-\frac{5}{12} \Delta t^3 u_{ttt}^n$ and therefore the local order of accuracy is $p = 2$ (i.e. one less than the order of the leading term). Note: this algorithm is actually the well-known 2nd order Adams-Bashforth method.

2. Now, apply this method to solve the problem,

   $$\dot{u} = -u^2,$$

   from $t = 0$ to 10 with initial condition $u(0) = 1$. Note: this problem was approximated in Lecture 2 using the forward Euler method and the midpoint method. To save yourself time, you may want to modify the Matlab script for the midpoint method instead of starting from scratch. Use the forward Euler method for the first timestep as was done for the midpoint method.

   Applying this method for timesteps of $\Delta t = 0.1$, 0.2, and 0.4, plot the solution and the error. Based on these results, show that the actual error is converging at a rate which agrees with the local truncation error analysis in the previous question.

   Solution: The results are shown in Figure 1. From the above analysis, the error is expected to decrease by a factor of 4 for every factor of 2 decrease in the timestep. This behavior can be confirmed by a close inspection of the results. For example, looking at the results at $t = 4$ shows that the error is 0.0065, 0.0017, and 0.0004 for decreasing Δt i.e. a factor of 4 decrease in the error.

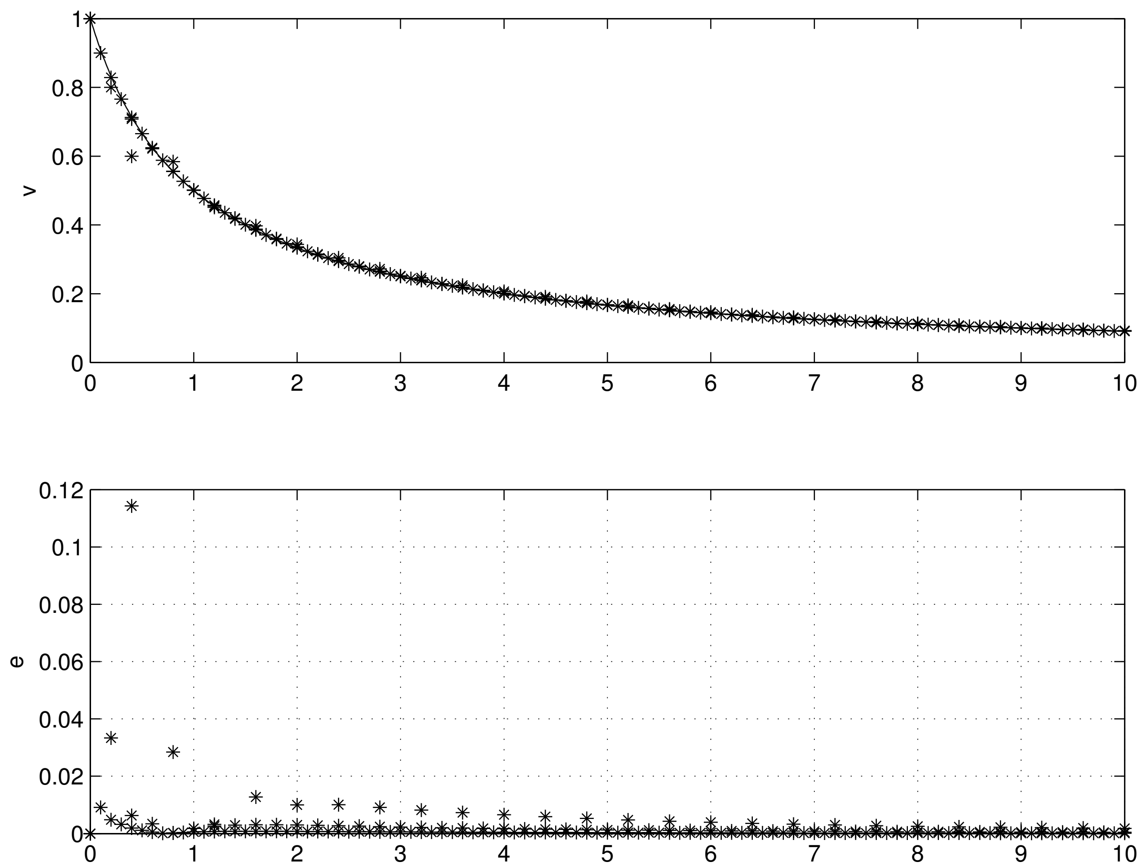

Figure 1: Second-order Adams-Bashforth approximation of $\dot{u} = -u^2$ with $u(0) = 1$ and $\Delta t = 0.1$, 0.2, and 0.4. Approximation (symbols) and exact solution (line) are shown in first plot. Error is shown in second plot.