16.901: Homework # 11
Solution

In this homework, you will modify the Matlab script **fem_dif1d_gq.m** from the Lecture Notes to use Gaussian quadrature for the evaluation of both the stiffness matrix and source term for the problem,

$$(kT_x)_x + q(x) = 0, \qquad k(x) = e^x, \quad q(x) = 50e^x,$$

with boundary conditions $T(\pm 1) = 100$. The script is available on the webpage under the Homework #11 link and already has Gaussian quadrature implemented for the source term. The exact solution to this problem is,

$$T(x) = 50 + 50\frac{e}{\sinh 1} - 50x - 50\frac{e^{-x}}{\sinh 1}.$$

WARNING: you will need to modify the exact solution in the **fem_dif1d_gq.m** script because it is for the Homework #10 problem (with constant k).

1. Modify the Matlab script using Gaussian quadrature for the stiffness matrix terms. Run the simulation for 5 elements and 10 elements using both one-point and two-point quadrature. For the homework, include the plots of the solutions for both quadrature rules as well as a hard copy of your completed script.

   Solution: As described in class, in element j, integral contributions to the j and $j+1$ weighted residuals are calculated and added to row j and $j+1$, respectively, of the stiffness matrix. For linear elements, these integrals are,

$$\begin{aligned}
\int_{x_j}^{x_{j+1}} \phi_{j_x} k\tilde{T}_x \, dx &= -\frac{a_{j+1} - a_j}{(\Delta x_j)^2} \int_{x_j}^{x_{j+1}} k \, dx, \\
\int_{x_j}^{x_{j+1}} \phi_{j+1_x} k\tilde{T}_x \, dx &= \frac{a_{j+1} - a_j}{(\Delta x_j)^2} \int_{x_j}^{x_{j+1}} k \, dx.
\end{aligned}$$

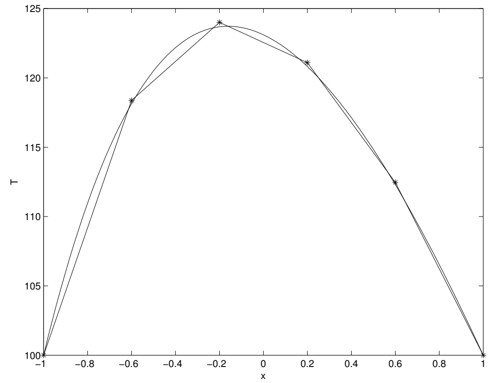
   The integrals of k are evaluated using Gaussian quadrature for which $g(\xi) = \frac{1}{2}\Delta x_j k(x(\xi))$. Thus, using quadrature the integrals will be evaluated as,

$$\begin{aligned}
\int_{x_j}^{x_{j+1}} \phi_{j_x} k\tilde{T}_x \, dx &= -\frac{a_{j+1} - a_j}{2\Delta x_j} \sum_{i=1}^{N_q} \alpha_i k(x(\xi_i)), \\
\int_{x_j}^{x_{j+1}} \phi_{j+1_x} k\tilde{T}_x \, dx &= \frac{a_{j+1} - a_j}{2\Delta x_j} \sum_{i=1}^{N_q} \alpha_i k(x(\xi_i)).
\end{aligned}$$

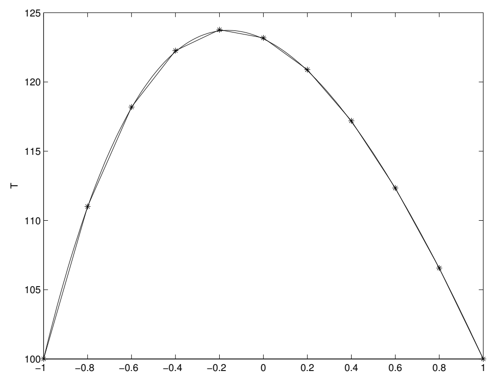
   The results are shown in Figures 1 and 2.

2. For both quadrature rules, what do you think the order of accuracy is for this finite element method? Justify your answer using the plots.

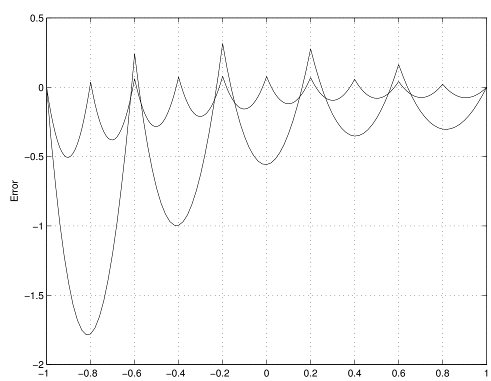
   Solution: As in the results from the previous homework (and in the notes), the accuracy appears to be second order with respect to the grid size for both quadrature rules since the error reduces by approximately a factor of four when the number of elements doubles. This can be seen in the error plots in the figures.



(a) $N = 5$ elements

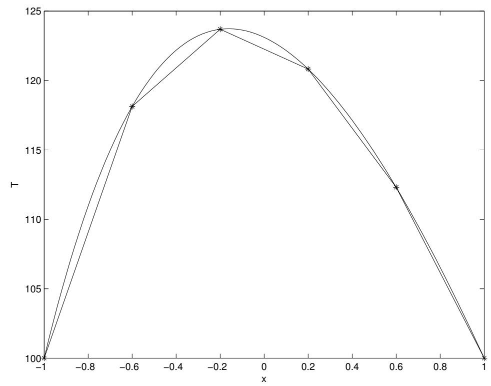


(b) $N = 10$ elements

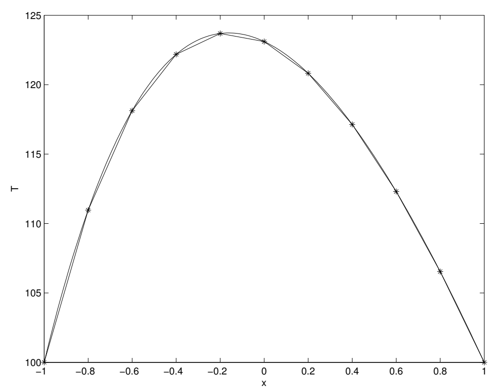


(c) Error

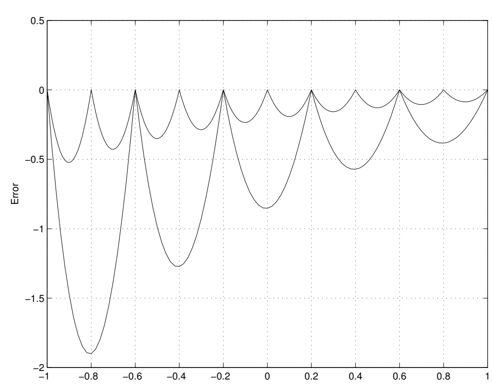
Figure 1: Comparison of finite element solution using $N_q = 1$ point Gaussian quadrature to exact solution.



(a) $N = 5$ elements



(b) $N = 10$ elements



(c) Error

Figure 2: Comparison of finite element solution using $N_q = 2$ point Gaussian quadrature to exact solution.