# 16.810

# Engineering Design and Rapid Prototyping

Lecture 3b

# CAE -Finite Element Method

Instructor(s)

Prof. Olivier de Weck

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Massachusetts Institute of Technology

# Numerical Methods

**Finite Element Method**

**Boundary Element Method**

**Finite Difference Method**

**Finite Volume Method**

**Meshless Method**

# What is the FEM?

**FEM: Method for numerical solution of field problems.**

**Description**

- **FEM cuts a structure into several elements (pieces of the structure).**
- **Then reconnects elements at "nodes" as if nodes were pins or drops of glue that hold elements together.**
- **This process results in a set of simultaneous algebraic equations.**

**Number of degrees-of-freedom (DOF)**

**Continuum:** **Infinite**

**FEM:** **Finite**

**(This is the origin of the name, Finite Element Method)**

# Fundamental Concepts (1)

**Many engineering phenomena can be expressed by "governing equations" and "boundary conditions"**

**Elastic problems**

**Thermal problems**

**Fluid flow**

**Electrostatics**

**etc.**

**Governing Equation**

**(Differential equation)**

$$L(\phi)+f=0$$

+

**Boundary Conditions**

$$B(\phi)+g=0$$

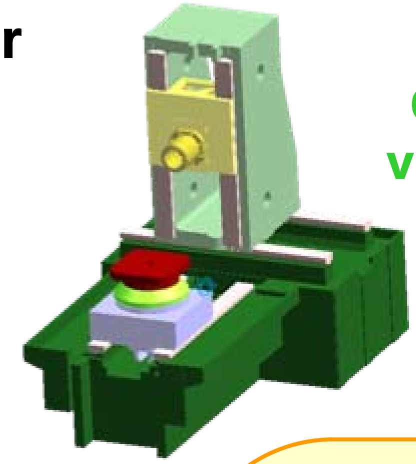
# Fundamental Concepts (2)

## Example: Vertical machining center

Elastic deformation

Thermal behavior

etc.



Geometry is very complex!

**Governing Equation:** $L(\phi)+f=0$

**Boundary Conditions:** $B(\phi)+g=0$

You know all the equations, but you cannot solve it by hand

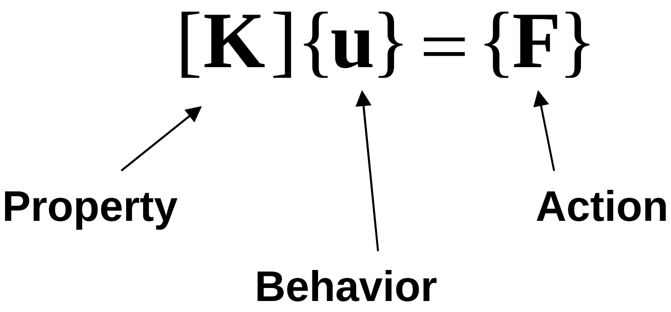
FEM

Approximate!

A set of simultaneous algebraic equations

$$[\mathbf{K}]\{\mathbf{u}\}=\{\mathbf{F}\}$$

# Fundamental Concepts (3)



$$[\mathbf{K}]\{\mathbf{u}\}=\{\mathbf{F}\} \quad \Rightarrow \quad \{\mathbf{u}\}=[\mathbf{K}]^{-1}\{\mathbf{F}\}$$

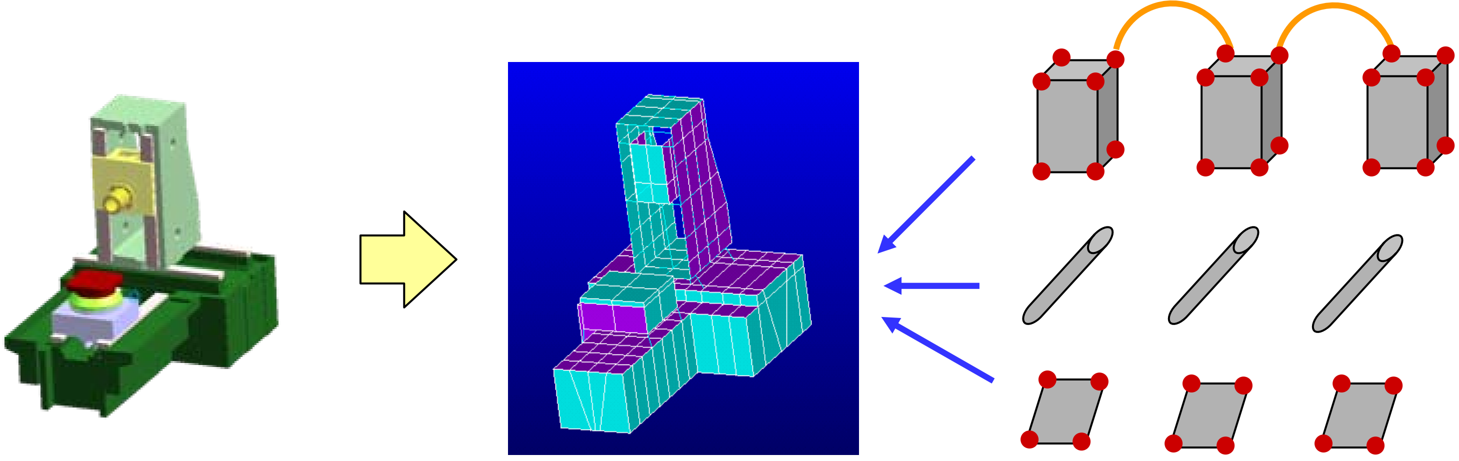
Unknown

| | Property $[\mathbf{K}]$ | Behavior $\{\mathbf{u}\}$ | Action $\{\mathbf{F}\}$ |
|---|---|---|---|
| **Elastic** | stiffness | displacement | force |
| **Thermal** | conductivity | temperature | heat source |
| **Fluid** | viscosity | velocity | body force |
| **Electrostatic** | Dielectric permittivity | electric potential | charge |

# Fundamental Concepts (4)

**It is very difficult to solve the algebraic equations for the entire domain**

- **Divide the domain into a number of small, simple elements**
  - **A field quantity is interpolated by a polynomial over an element**
    - **Adjacent elements share the DOF at connecting nodes**

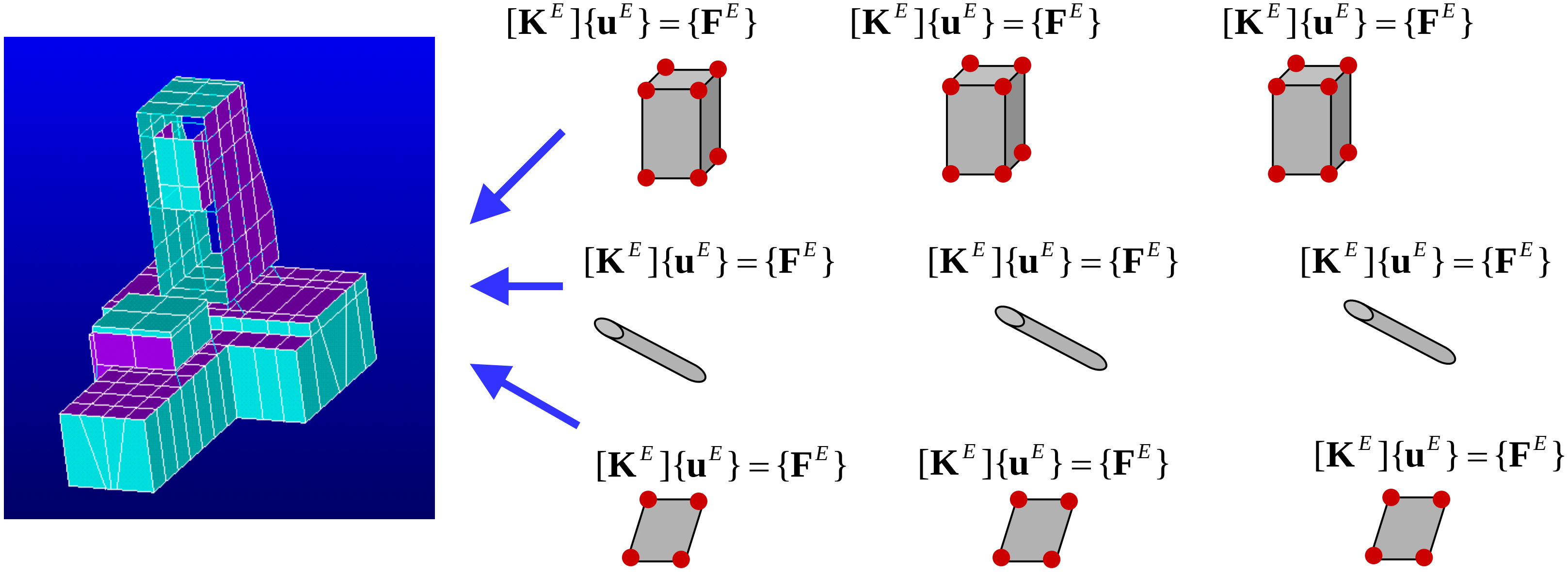


Finite element: Small piece of structure

# Fundamental Concepts (5)

**Obtain the algebraic equations for each element (this is easy!)**

⇨ **Put all the element equations together**



$[\mathbf{K}^E]\{\mathbf{u}^E\}=\{\mathbf{F}^E\}$ $[\mathbf{K}^E]\{\mathbf{u}^E\}=\{\mathbf{F}^E\}$ $[\mathbf{K}^E]\{\mathbf{u}^E\}=\{\mathbf{F}^E\}$

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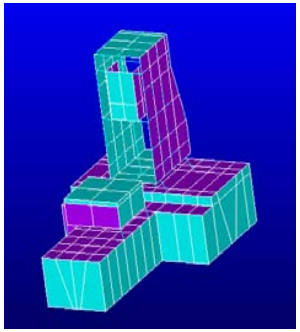
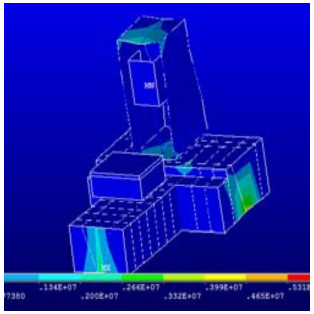
$$[\mathbf{K}]\{\mathbf{u}\}=\{\mathbf{F}\}$$

# Fundamental Concepts (6)

**Solve the equations, obtaining unknown variables at nodes.**

$$[\mathbf{K}]\{\mathbf{u}\}=\{\mathbf{F}\} \quad \Longrightarrow \quad \{\mathbf{u}\}=[\mathbf{K}]^{-1}\{\mathbf{F}\}$$



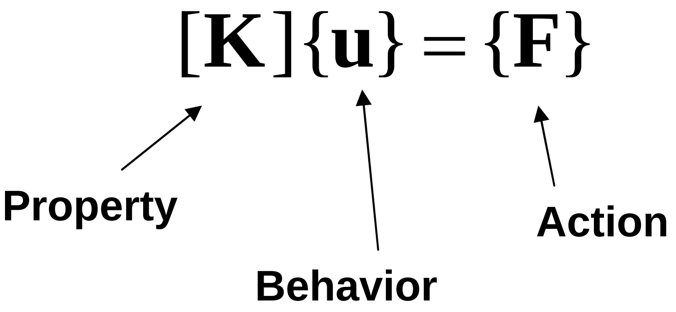


# Concepts - Summary

- FEM uses the concept of piecewise polynomial interpolation.

- By connecting elements together, the field quantity becomes interpolated over the entire structure in piecewise fashion.

- A set of simultaneous algebraic equations at nodes.



$$[\mathbf{K}]\{\mathbf{u}\}=\{\mathbf{F}\}$$

**Property**

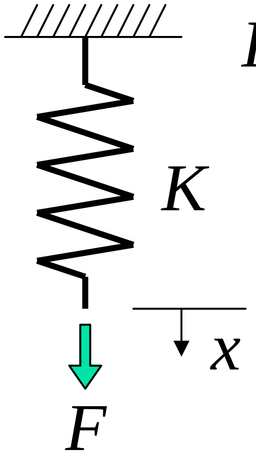
**Behavior**

**Action**

*K: Stiffness matrix*

*x: Displacement*

*F: Load*



$$Kx = F$$

# Brief History

- The term finite element was first coined by Clough in 1960. In the early 1960s, engineers used the method for approximate solutions of problems in stress analysis, fluid flow, heat transfer, and other areas.

- The first book on the FEM by Zienkiewicz and Chung was published in 1967.

- In the late 1960s and early 1970s, the FEM was applied to a wide variety of engineering problems.

- Most commercial FEM software packages originated in the 1970s. (Abaqus, Adina, Ansys, etc.)

- Klaus-Jurgen Bathe in ME at MIT

*Reference [2]*

# Advantages of the FEM

**Can readily handle very complex geometry:**

- The heart and power of the FEM

**Can handle a wide variety of engineering problems**

| | | |
|---|---|---|
| - Solid mechanics | - Dynamics | - Heat problems |
| - Fluids | - Electrostatic problems | |

**Can handle complex restraints**

- Indeterminate structures can be solved.

**Can handle complex loading**

- Nodal load (point loads)
- Element loads - distributed (pressure, thermal, inertial forces)
- Time or frequency dependent loading

# Disadvantages of the FEM

**A general closed-form solution, which would permit one to examine system response to changes in various parameters, is not produced.**

**The FEM obtains only "approximate" solutions.**

**The FEM has "inherent" errors.**

**Mistakes by users can remain undetected.**

# Typical FEA Procedure by Commercial Software

**User** — **Preprocess** — **Build a FE model**

**Computer** — **Process** — **Conduct numerical analysis**

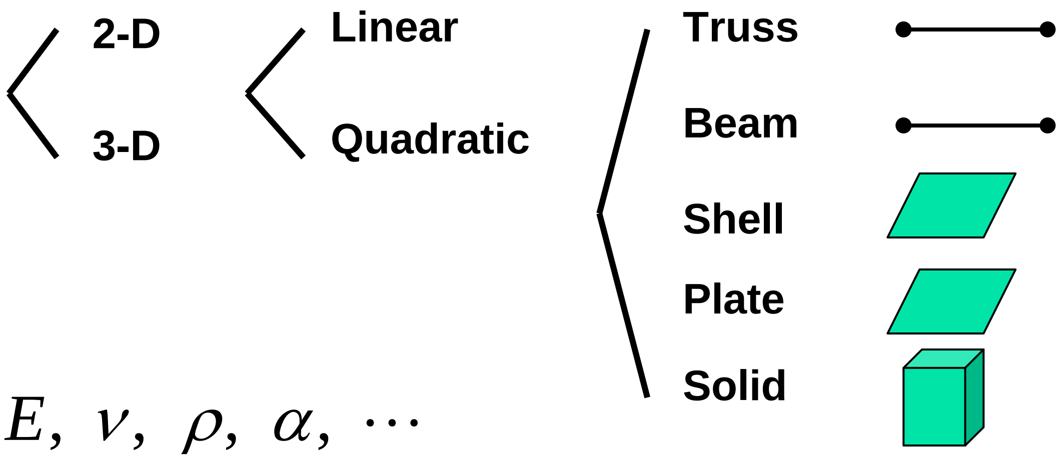
**User** — **Postprocess** — **See results**

# Preprocess (1)

**[1] Select analysis type**

- Structural Static Analysis
- Modal Analysis
- Transient Dynamic Analysis
- Buckling Analysis
- Contact
- Steady-state Thermal Analysis
- Transient Thermal Analysis

**[2] Select element type**



2-D
3-D

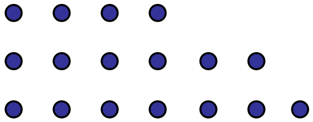
Linear
Quadratic

Truss
Beam
Shell
Plate
Solid

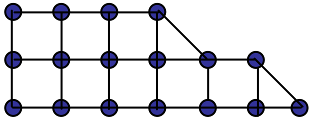
**[3] Material properties** $E,\ \nu,\ \rho,\ \alpha,\ \cdots$

# Preprocess (2)

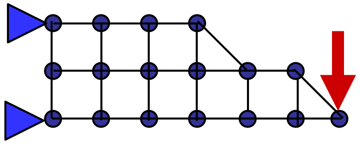
[4] Make nodes



[5] Build elements by assigning connectivity



[6] Apply boundary conditions and loads



# Process and Postprocess

[7] Process

- Solve the boundary value problem

[8] Postprocess

- See the results

**Displacement**

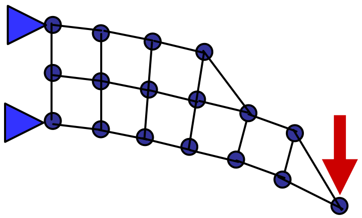
**Stress**

**Strain**

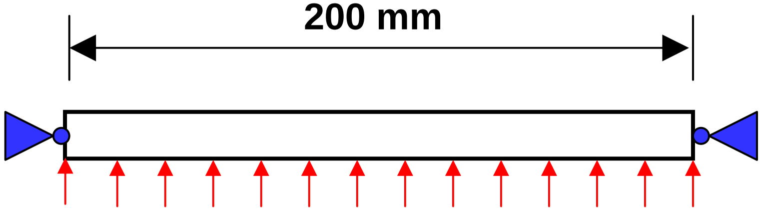
**Natural frequency**

**Temperature**

**Time history**



# Responsibility of the user



1 ms pressure pulse

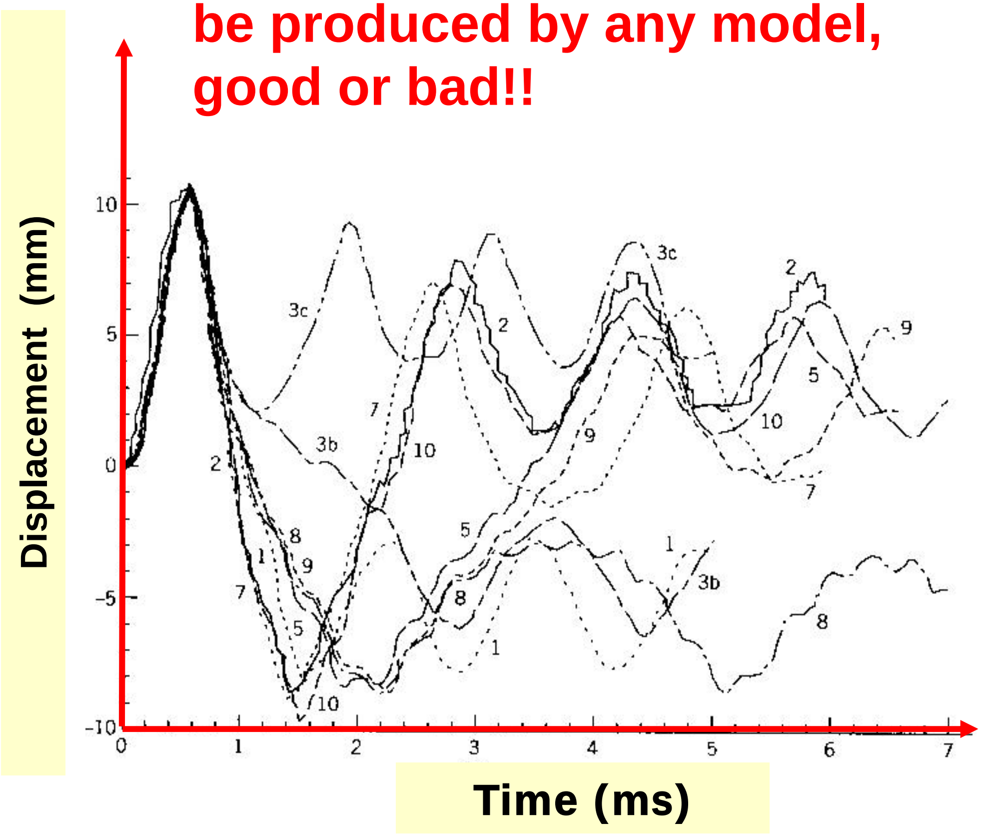
**BC:** Hinged supports

**Load:** Pressure pulse

**Unknown:** Lateral mid point displacement in the time domain

**Results obtained from ten reputable FEM codes and by users regarded as expert.***

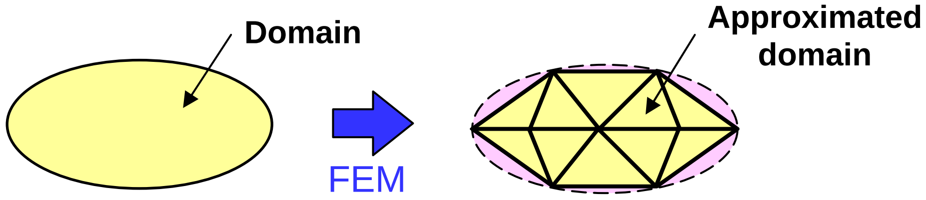
**Fancy, colorful contours can be produced by any model, good or bad!!**



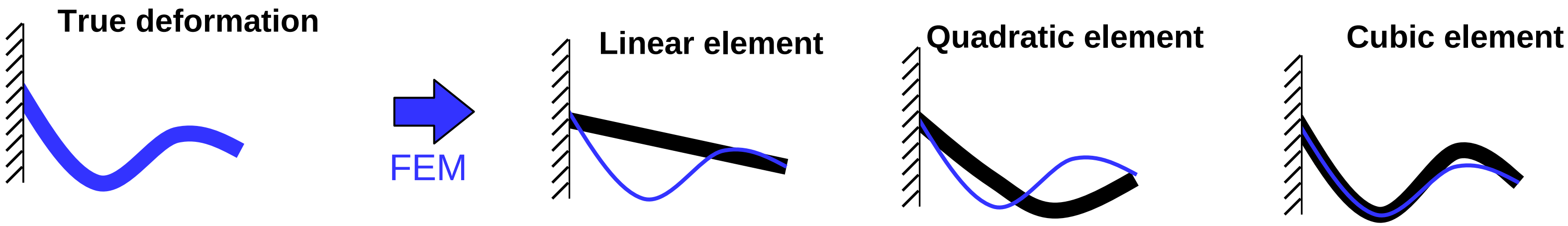
* *R. D. Cook, Finite Element Modeling for Stress Analysis, John Wiley & Sons, 1995*

# Errors Inherent in FEM Formulation

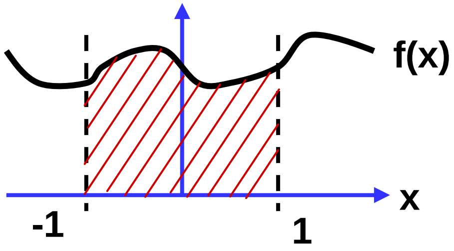
- Geometry is simplified.



- Field quantity is assumed to be a polynomial over an element. (which is not true)



- Use very simple integration techniques (Gauss Quadrature)



$$\text{Area:} \quad \int_{-1}^{1} f(x)\,dx \approx f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$$

# 2-D vs. 3-D

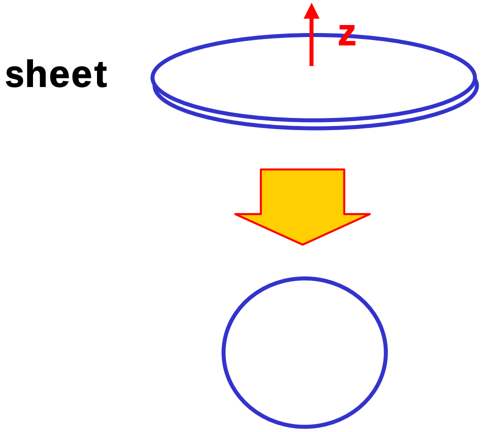
**In reality, everything is 3-D.**

**But some problems can be simplified to 2-D (in structures, plane stress and plane strain).**

## Plane Stress

$$\sigma_z = 0$$

$$\text{thickness} \approx 0$$



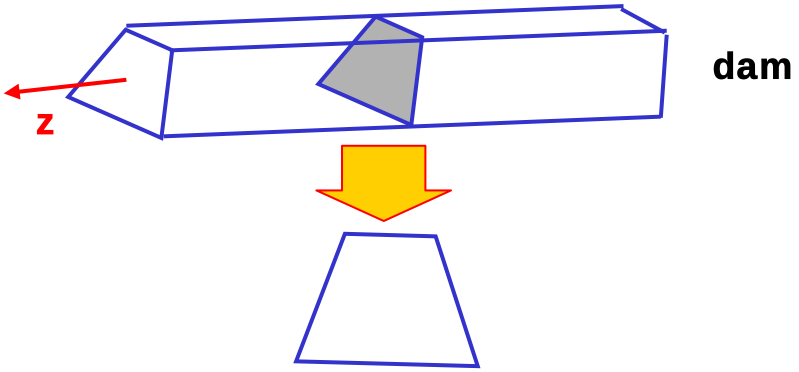
3-D

2-D

## Plane Strain

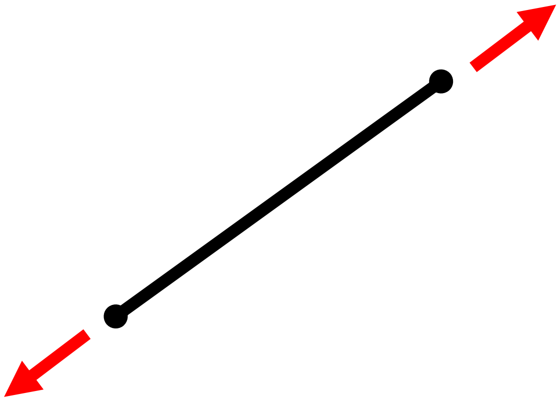
$$\varepsilon_z = 0$$

$$\text{thickness} \approx \infty$$



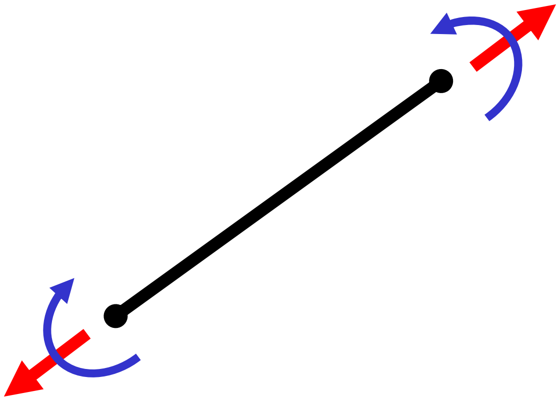
# Truss vs. Beam

## Truss



**Only supports axial loads**

## Beam



**Supports axial loads and bending loads**

# Errors Inherent in Computing

- The computer carries only a finite number of digits.

$$\text{e.g.)}\quad \sqrt{2} = 1.41421356, \qquad \pi = 3.14159265$$

- Numerical Difficulties

**e.g.) Very large stiffness difference**

$$k_1 >> k_2 \quad , \quad k_2 \approx 0$$

$$[(k_1 + k_2) - k_2]u_2 = P \; \Rightarrow u_2 = \frac{P}{k_2} \approx \frac{P}{0}$$

# Mistakes by Users

- Elements are of the wrong type
  e.g) Shell elements are used where solid elements are needed

- Distorted elements

- Supports are insufficient to prevent all rigid-body motions

- Inconsistent units (e.g.  E=200 GPa, Force = 100 lbs)

- Too large stiffness differences → Numerical difficulties

# References

Glaucio H. Paulino, *Introduction to FEM (History, Advantages and Disadvantages)*, http://cee.uiuc.edu/paulino/index.htm

Robert Cook et al., *Concepts and Applications of Finite Element Analysis*, John Wiley & Sons, 1989

Robert Cook, *Finite Element Modeling For Stress Analysis*, John Wiley & Sons, 1995

*Introduction to Finite Element Method*, http://210.17.155.47 (in Korean)

J. Tinsley Oden et al., *Finite Elements – An Introduction*, Prentice Hall, 1981