ERROR ANALYSIS
(UNCERTAINTY ANALYSIS)

16.621 Experimental Projects Lab I

TOPICS TO BE COVERED

- **Why do error analysis?**
- **If we don't ever know the true value, how do we estimate the error in the true value?**
- **Error propagation in the measurement chain**
  - **How do errors combine? (How do they behave in general?)**
  - **How do we do an end-to-end uncertainty analysis?**
  - **What are ways to mitigate errors?**
- **A hypothetical dilemma (probably nothing to do with anyone in the class)**
  - **When should I throw out some data that I don't like?**
  - **Answer: NEVER, but there are reasons to throw out data**

- **Backup slides: an example of an immense amount of money and effort directed at error analysis and mitigation - jet engine testing**

ERROR AND UNCERTAINTY

- **In engineering the word "error", when used to describe an aspect of measurement does not necessarily carry the connotation of mistake or blunder (although it can!)**

- **Error in a measurement means the inevitable uncertainty that attends all measurements**

- **We cannot avoid errors in this sense**

- **We can ensure that they are as small as reasonably possible and that we have a reliable estimate of how small they are**

[Adapted from Taylor, J. R, *An Introduction to Error Analysis; The Study of Uncertainties in Physical Measurements*]

USES OF UNCERTAINTY ANALYSIS (I)

- **Assess experimental procedure including identification of potential difficulties**
  - **Definition of necessary steps**
  - **Gaps**

- **Advise what procedures need to be put in place for measurement**

- **Identify instruments and procedures that control accuracy and precision**
  - **Usually one, or at most a small number, out of the large set of possibilities**

- **Inform us when experiment cannot meet desired accuracy**

USES OF UNCERTAINTY ANALYSIS (II)

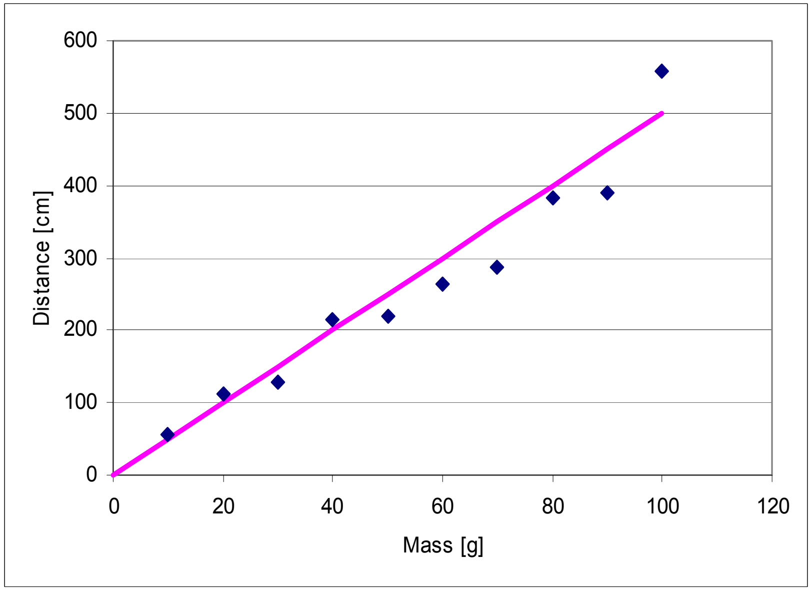
- Provide the only known basis for deciding whether:
  - Data agrees with theory
  - Tests from different facilities (jet engine performance) agree
  - Hypothesis has been appropriately assessed (resolved)
  - Phenomena measured are real

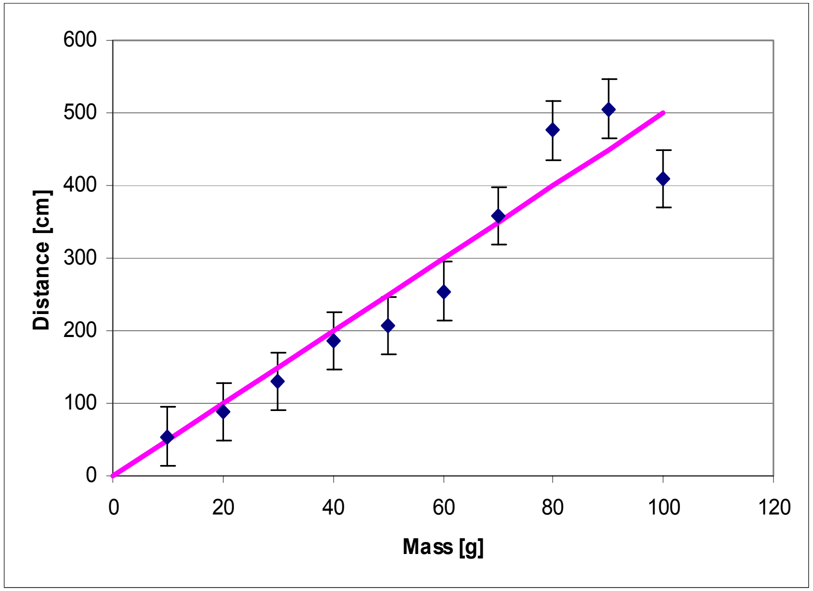
- Provide basis for defining whether a closure check has been achieved
  - Is continuity satisfied (does the same amount of mass go in as goes out?)
  - Is energy conserved?

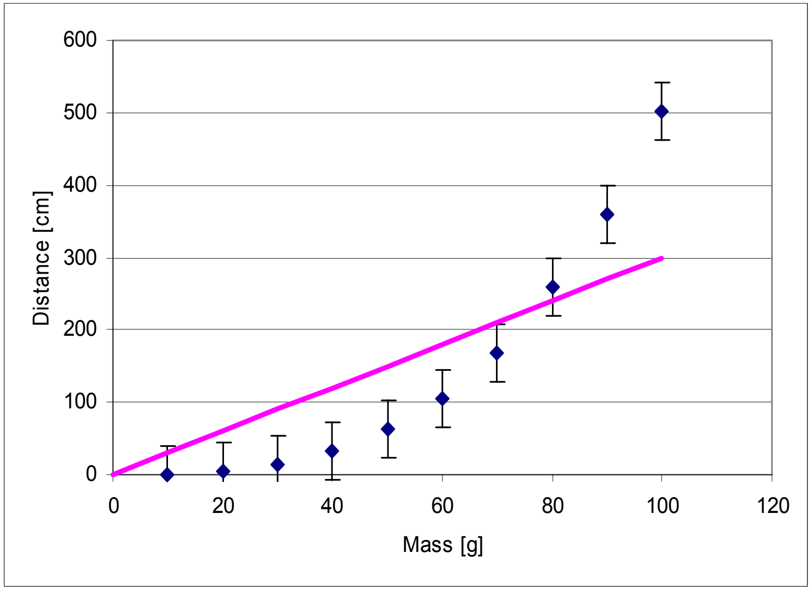
- Provide an integrated grasp of how to conduct the experiment

[Adapted from Kline, S. J., 1985, "The Purposes of Uncertainty Analysis", ASME *J. Fluids Engineering*, pp. 153-160]

UNCERTAINTY ESTIMATES AND HYPOTHESIS ASSESSMENT







HOW DO WE DEAL WITH NOT KNOWING THE TRUE VALUE?

- **In “all” real situations we don’t know the true value we are looking for**

- **We need to decide how to determine the best representation of this from our measurements**

- **We need to decide what the uncertainty is in our best representation**

AN IMPLICATION OF NOT KNOWING THE TRUE VALUE

- **We easily divided errors into precision (bias) errors and random errors when we knew what the value was**

- **The target practice picture in the next slide is an example**

- **How about if we don’t know the true value? Can we, by looking at the data in the slide after this, say that there are bias errors?**

- **How do we know if bias errors exist or not?**

A TEAM EXERCISE

- List the variables you need to determine in order to carry out your hypothesis assessment
- What uncertainties do you foresee? (Qualitative description)
- Are you more concerned about bias errors or random errors?
- What level of uncertainty in the final result do you need to assess your hypothesis in a rigorous manner?
- Can you make an estimate of the level of the uncertainty in the final result?
  - If so, what is it?
  - If not, what additional information do you need to do this?

HOW DO WE COMBINE ERRORS?

- **Suppose we measure quantity X with an error of dx and quantity Y with an error of dy**

- **What is the error in quantity Z if:**
  - **Z = AX where A is a numerical constant such as π?**
  - **Z = X + Y?**
  - **Z = X - Y?**
  - **Z = XY?**
  - **Z = X/Y?**
  - **Z is a general function of many quantities?**

ERRORS IN THE FINAL QUANTITY

- Z = X + Y
- Linear combination
  - $Z + |dz| = X + |dx| + Y + |dy|$
  - Error in Z is $|dz| = |dx| + |dy|$ BUT this is worst case
- For random errors we could have
  - $|dz| = |dx| - |dy|$

    or $|dy| - |dx|$
  - These errors are much smaller
- In general if different errors are not correlated, are independent, the way to combine them is

$$\boxed{dz = \sqrt{dx^2 + dy^2}}$$

- This is true for random and bias errors

THE CASE OF Z = X - Y

- **Suspect Z = X - Y is a number much smaller than X or Y**

$$dz = \sqrt{dx^2 + dy^2}$$

- **Say** $\frac{dx}{X} = \frac{dy}{Y} = \varepsilon$ **(say 2%)**

- $\frac{dz}{Z} = \frac{\sqrt{2}\,dx}{X - Y}$ **may be much larger than** $\frac{dx}{X}$

- **MESSAGE ==> Avoid taking the difference of two numbers of comparable size**

ESTIMATES FOR THE TRUE VALUE AND THE ERROR

- **Is there a "best" estimate of the true value of a quantity?**
- **How do I find it?**
- **How do I estimate the random error?**
- **How do I estimate the bias error?**

SOME "RULES" FOR ESTIMATING RANDOM ERRORS AND TRUE VALUE

- **An internal estimate can be given by repeat measurements**
- **Random error is generally of same size as standard deviation (root mean square deviation) of measurements**
- **Mean of repeat measurements is best estimate of true value**
- **Standard deviation of the mean (random error) is smaller than standard deviation of a single measurement by**

$$1/\sqrt{\text{Number of measurements}}$$

- **To increase precision by 10, you need 100 measurements**

GENERAL RULE FOR COMBINATION OF ERRORS

- **If $Z = F(X_1, X_2, X_3, X_4)$ is quantity we want**
- **The error in Z, dz, is given by our rule from before**
- **So, if the error F due to X_1 can be estimated as**

Error in X_1

$$dF_1 = \frac{\partial F}{\partial X_1} dx_1 \qquad \text{and so on}$$

Influence coeff.

- $$dz = \sqrt{\left(\frac{\partial F}{\partial X_1}\right)^2 dx_1^2 + \left(\frac{\partial F}{\partial X_2}\right)^2 dx_2^2 + \cdots \left(\frac{\partial F}{\partial X_n}\right)^2 dx_n^2}$$

- **The important consequence of this is that generally one or few of these factors is the main player and <u>others can be ignored</u>**

DISTRIBUTION OF RANDOM ERRORS

- **A measurement subject to many small random errors will be distributed "normally"**

- **Normal distribution is a Gaussian**

- **If x is a given measurement and X is the true value**

$$\text{Gaussian or normal distribution} = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\left(x - X^2\right)/2\sigma^2}$$

- **σ is the <u>standard deviation</u>**

A REVELATION

- The universal gas constant is

  accepted R = 8.31451 ±0.00007 J/mol K

- This is not a true value but can be "accepted" as one

ONE ADDITIONAL ASPECT OF COMBINING ERRORS

- **We have identified two different types of errors, bias (systematic) and random**
  - **Random errors can be assessed by repetition of measurements**
  - **Bias errors cannot; these need to be estimated using external information (mfrs. specs., your knowledge)**

- **How should the two types of errors be combined?**
  - **One practice is to treat each separately using our rule, and then report the two separately at the end**
  - **One other practice is to combine them as “errors”**
- **Either seems acceptable, as long as you show that you are going to deal (have dealt) with both**

REPORTING OF MEASUREMENTS

- **Experimental uncertainties should almost always be rounded to one significant figure**

- **The last significant figure in any stated answer should usually be of the same order of magnitude (in the same decimal position) as the uncertainty**

[from Taylor, J., *An Introduction to Error Analysis*]

COMMENTS ON REJECTION OF DATA

- **Should you reject (delete) data?**
- **Sometimes on measurement appears to disagree greatly with all others. How do we decide:**
  - **Is this significant?**
  - **Is this a mistake?**
- **One criteria (Chauvenaut's criteria) is as follows**
  - **Suppose that errors are normally distributed**
  - **If measurement is more than M standard deviations (say 3), probability is < 0.003 that measurement should occur**
  - **Is this improbable enough to throw out measurement?**
- **The decision of "ridiculous improbability" [Taylor, 1997] is up to the investigator, but it allows the reader to understand the basis for the decision**
  - **If beyond this range, delete the data**

A CAVEAT ON REJECTION OF DATA

- **If more than one measurement is different, it may be that something is really happening that has not been envisioned, *e.g.*, discovery of radon**

- **You may not be controlling all the variables that you need to**

- **Bottom line: Rigorous uncertainty analysis can give rationale to decide what data to pay attention to**

SUMMARY

- Both the number and the fidelity of the number are important in a measurement
- We considered two types of uncertainties, bias (or systematic errors) and random errors
- Uncertainty analysis addresses fidelity and is used in different phases of an experiment, from initial planning to final reporting
  - Attention is needed to ensure uncertainties do not invalidate your efforts
- In propagating uncorrelated errors from individual measurement to final result, use the square root of the sums of the squares of the errors
  - There are generally only a few main contributors (sometimes one) to the overall uncertainty which need to be addressed
- Uncertainty analysis is a critical part of “real world” engineering projects

SOME REFERENCES I HAVE FOUND USEFUL

- Baird, D. C., 1962, *Experimentation: An Introduction to Measurement Theory and Experiment*, Prentice-Hall, Englewood Cliffs, NJ

- Bevington, P. R, and Robinson, D. K., 1992, *Data Reduction and Error Analysis for the Physical Sciences*, McGraw-Hill, New York, NY

- Lyons, L., 1991, *A Practical Guide to Data Analysis for Physical Science Students*, Cambridge University Press, Cambridge, UK

- Rabinowicz, E, 1970, *An Introduction to Experimentation*, Addison-Wesley, Reading, MA

- Taylor, J. R., 1997, *An Introduction to Error Analysis*, University Science Books, Sauselito, CA

BACKUP EXAMPLE: MEASUREMENT OF JET ENGINE PEFORMANCE

- **We want to measure Thrust, Airflow, and Thrust Specific Fuel Consumption (TSFC)**
  - **Engine program can be $1B or more, take three years or more**
  - **Engine companies give guarantees in terms of fuel burn**
  - **Engine thrust needs to be correct or aircraft can't take off in the required length**
  - **Airflow fundamental in diagnosing engine performance**
  - **These are basic and essential measures**
- **How do we measure thrust?**
- **How do we measure airflow?**
- **How do we measure fuel flow?**

THRUST STANDS

- In practice, thrust is measured with load cells
- The engines, however, are often part of a complex test facility and are connected to upstream ducting
- There are thus certain systematic errors which need to be accounted for
- The level of uncertainty in the answer is desired to be less than one per cent
- There are a lot of corrections to be made to the raw data (measured load) to give the thrust

TEST STAND-TO-TEST STAND DIFFERENCES

- **Want to have a consistent view of engine performance no matter who quotes the numbers**
- **This means that different test stands must be compared to see the differences**
- **Again, this is a major exercise involving the running of a jet engine in different locations under specified conditions**
- **The next slide shows the level of differences in the measurements**