Lecture #4

16.61 Aerospace Dynamics

- Extension to multiple intermediate frames (two)

Introduction

- We started with one frame (B) rotating $\vec{\omega}$ and accelerating $\dot{\vec{\omega}}$ with respect to another (I), and obtained the following expression for the absolute acceleration
$$\ddot{\vec{r}}^{\,I} = \ddot{\vec{r}}_{cm}^{\,I} + \ddot{\vec{\rho}}^{\,B} + 2\vec{\omega} \times \dot{\vec{\rho}}^{\,B} + \dot{\vec{\omega}}^{\,I} \times \vec{\rho} + \vec{\omega} \times (\vec{\omega} \times \vec{\rho})$$
of a point located at
$$\vec{r} = \vec{r}_{cm} + \vec{\rho}$$

- However, in many cases there are often several intermediate frames that have to be taken into account.

- Consider the situation in the figure:
  - Two frames that are moving, rotating, accelerating with respect to each other and the inertial reference frame.
  - Assume that ${}^2\vec{\omega}$ and ${}^2\dot{\vec{\omega}}^{\,1}$ are given with respect to the first intermediate frame 1.
    - ◇ The left superscript here simply denotes a label – there are two $\vec{\omega}$'s to consider in this problem.

- Note that the position of point P with respect to the origin of frame 1 is given by
$${}^{P1}\vec{r} = {}^{2}\vec{r} + {}^{3}\vec{r}$$
and the position of point P with respect to the origin of the inertial frame is given by
$${}^{PI}\vec{r} = {}^{1}\vec{r} + {}^{2}\vec{r} + {}^{3}\vec{r} \equiv {}^{1}\vec{r} + {}^{P1}\vec{r}$$

- We want ${}^{PI}\dot{\vec{r}}^{\,I}$ and ${}^{PI}\ddot{\vec{r}}^{\,I}$

- The approach is to find the motion of P with respect to frame 1.

$$\begin{aligned} {}^{P1}\dot{\vec{r}}^{\,1} &= {}^{2}\dot{\vec{r}}^{\,1} + {}^{3}\dot{\vec{r}}^{\,1} \equiv {}^{P1}\vec{v} \\ &= {}^{2}\dot{\vec{r}}^{\,1} + ({}^{3}\dot{\vec{r}}^{\,2} + {}^{2}\vec{\omega} \times {}^{3}\vec{r}\,) \end{aligned}$$

and

$$\begin{aligned} {}^{P1}\ddot{\vec{r}}^{\,1} &= {}^{2}\ddot{\vec{r}}^{\,1} + {}^{3}\ddot{\vec{r}}^{\,1} \equiv {}^{P1}\vec{a} \\ &= {}^{2}\ddot{\vec{r}}^{\,1} + ({}^{3}\ddot{\vec{r}}^{\,2} + 2({}^{2}\vec{\omega} \times {}^{3}\dot{\vec{r}}^{\,2}) + {}^{2}\dot{\vec{\omega}}^{\,1} \times {}^{3}\vec{r} + {}^{2}\vec{\omega} \times ({}^{2}\vec{\omega} \times {}^{3}\vec{r}\,)) \end{aligned}$$

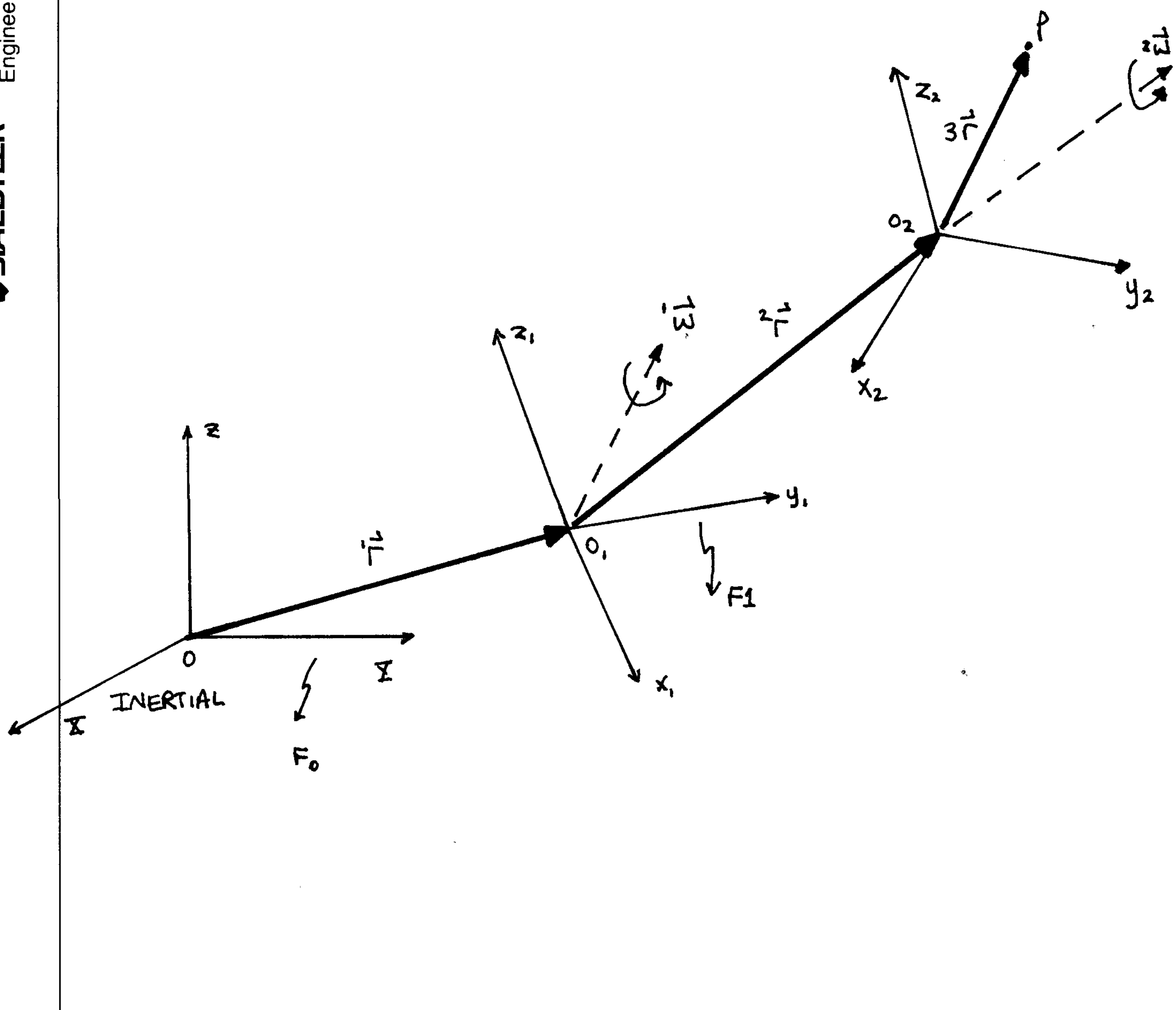
- While the notation is a bit laborious, there is nothing new here – this is just the same case we have looked at before with one frame moving with respect to another.
  - Steps above were done ignoring the motion of frame 1 altogether.

- Now consider what happens when we include the fact that frame 1 is moving with respect to the inertial frame I. Again:

$${}^{PI}\vec{r} = {}^{1}\vec{r} + {}^{2}\vec{r} + {}^{3}\vec{r} \equiv {}^{1}\vec{r} + {}^{P1}\vec{r}$$

Now compute the desired velocity:

$$\begin{aligned} {}^{PI}\dot{\vec{r}}^{\,I} &= {}^{1}\dot{\vec{r}}^{\,I} + {}^{P1}\dot{\vec{r}}^{\,I} \equiv {}^{PI}\vec{v} \\ &= {}^{1}\dot{\vec{r}}^{\,I} + {}^{P1}\dot{\vec{r}}^{\,1} + ({}^{1}\vec{\omega} \times {}^{P1}\vec{r}\,) \\ &= {}^{1}\dot{\vec{r}}^{\,I} + \left[{}^{2}\dot{\vec{r}}^{\,1} + ({}^{3}\dot{\vec{r}}^{\,2} + {}^{2}\vec{\omega} \times {}^{3}\vec{r}\,)\right] + {}^{1}\vec{\omega} \times ({}^{2}\vec{r} + {}^{3}\vec{r}\,) \\ &= {}^{1}\dot{\vec{r}}^{\,I} + {}^{2}\dot{\vec{r}}^{\,1} + {}^{3}\dot{\vec{r}}^{\,2} + {}^{1}\vec{\omega} \times {}^{2}\vec{r} + ({}^{1}\vec{\omega} + {}^{2}\vec{\omega}) \times {}^{3}\vec{r} \end{aligned}$$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And acceleration:

$$
\begin{aligned}
{}^{PI}\ddot{\vec{r}}^{\,I} &= {}^{1}\ddot{\vec{r}}^{\,I} + {}^{P1}\ddot{\vec{r}}^{\,I} \equiv {}^{PI}\vec{a} \\
&= {}^{1}\ddot{\vec{r}}^{\,I} + \frac{d^I}{dt}\left[{}^{P1}\dot{\vec{r}}^{\,I}\right] \\
&= {}^{1}\ddot{\vec{r}}^{\,I} + \frac{d^I}{dt}\left[{}^{P1}\dot{\vec{r}}^{\,1} + ({}^{1}\vec{\omega} \times {}^{P1}\vec{r}\,)\right] \\
&= {}^{1}\ddot{\vec{r}}^{\,I} + \left({}^{P1}\ddot{\vec{r}}^{\,1} + {}^{1}\vec{\omega} \times {}^{P1}\dot{\vec{r}}^{\,1}\right) + {}^{1}\dot{\vec{\omega}}^{\,I} \times {}^{P1}\vec{r} \\
&\qquad + {}^{1}\vec{\omega} \times \left[{}^{P1}\dot{\vec{r}}^{\,1} + {}^{1}\vec{\omega} \times {}^{P1}\vec{r}\,\right]
\end{aligned}
$$

- Now substitute and condense.

$$
\begin{aligned}
{}^{PI}\ddot{\vec{r}}^{\,I} &= {}^{1}\ddot{\vec{r}}^{\,I} + {}^{P1}\ddot{\vec{r}}^{\,1} + 2({}^{1}\vec{\omega} \times {}^{P1}\dot{\vec{r}}^{\,1}) + {}^{1}\dot{\vec{\omega}}^{\,I} \times {}^{P1}\vec{r} + {}^{1}\vec{\omega} \times ({}^{1}\vec{\omega} \times {}^{P1}\vec{r}\,) \\
&= {}^{1}\ddot{\vec{r}}^{\,I} + \\
&\quad \left\{{}^{2}\ddot{\vec{r}}^{\,1} + ({}^{3}\ddot{\vec{r}}^{\,2} + 2({}^{2}\vec{\omega} \times {}^{3}\dot{\vec{r}}^{\,2}) + {}^{2}\dot{\vec{\omega}}^{\,1} \times {}^{3}\vec{r} + {}^{2}\vec{\omega} \times ({}^{2}\vec{\omega} \times {}^{3}\vec{r}\,))\right\} \\
&\quad +2\left({}^{1}\vec{\omega} \times \left\{{}^{2}\dot{\vec{r}}^{\,1} + {}^{3}\dot{\vec{r}}^{\,2} + {}^{2}\vec{\omega} \times {}^{3}\vec{r}\,\right\}\right) \\
&\quad +{}^{1}\dot{\vec{\omega}}^{\,I} \times \left\{{}^{2}\vec{r} + {}^{3}\vec{r}\,\right\} + {}^{1}\vec{\omega} \times ({}^{1}\vec{\omega} \times \left\{{}^{2}\vec{r} + {}^{3}\vec{r}\,\right\}) \\
&= {}^{1}\ddot{\vec{r}}^{\,I} + {}^{2}\ddot{\vec{r}}^{\,1} + {}^{3}\ddot{\vec{r}}^{\,2} + 2([{}^{1}\vec{\omega} + {}^{2}\vec{\omega}] \times {}^{3}\dot{\vec{r}}^{\,2}) + 2({}^{1}\vec{\omega} \times {}^{2}\dot{\vec{r}}^{\,1}) \\
&\quad +{}^{1}\dot{\vec{\omega}}^{\,I} \times {}^{2}\vec{r} + [{}^{1}\dot{\vec{\omega}}^{\,I} + {}^{2}\dot{\vec{\omega}}^{\,1}] \times {}^{3}\vec{r} \\
&\quad +{}^{2}\vec{\omega} \times ({}^{2}\vec{\omega} \times {}^{3}\vec{r}\,) + {}^{1}\vec{\omega} \times ({}^{1}\vec{\omega} \times [{}^{2}\vec{r} + {}^{3}\vec{r}\,]) + 2\,{}^{1}\vec{\omega} \times ({}^{2}\vec{\omega} \times {}^{3}\vec{r}\,)
\end{aligned}
$$

- This final expression looks messy, but it is really just two nested versions of what we have seen before.
  - The nesting can continue to more levels and can be automated.

- Note that this type of expression is very easy to get wrong if not done carefully and systematically.
  - The hardest term to capture here is the $2\,{}^1\vec{\omega} \times ({}^2\vec{\omega} \times {}^3\vec{r}\,)$ which comes from the $2({}^1\vec{\omega} \times {}^{P1}\dot{\vec{r}}\,^1)$ term.
  - ${}^{P1}\dot{\vec{r}}\,^1$ is the relative velocity of P with respect to the origin of frame 1 as seen by an observer in the rotating frame 1.

- Example: acceleration of the tip of the tail rotor on a helicopter:
  - The helicopter body is rotating.
  - The tail rotor is rotating as well, but the base is attached to the body.