Relationship Between Drifts and the Fluid Picture

In the previous sections we have progressively collected a variety of effects which each can yield a "drift current": we have an Inertia Current, a ∇B-current and, if only one polarity is magnetized, and $E \times B$ current, or Hall current. Additionally, we had earlier pointed out that, even if the Guiding Centers do not drift, an inhomogeneous distribution of Larmor rotations gives rise to a Magnetization Current.

On the other hand, very often we will think of the plasma as a fluid characterized by a density n, a mean velocity $\vec{u}$, a current density $\vec{j}$ and a temperature T (at least one such fluid for each of the particle species present). What is the connection between these two pictures? We will see that, if all drifts plus magnetization currents are properly accounted for, the "fluid" does satisfy the fluid momentum balance, as it should. On the other hand, any physical conclusion drawn from only one, or an incomplete subset of drifts can easily be very misleading, because the various effects can cancel each other in complex ways. In this sense, although the drift analysis is very useful in clarifying the mechanisms at work (and can, for example, help devise effective plasma simulation techniques), the fluid picture is safer, because all those cancellations are built-in from the start. One limitation of the fluid picture is that the resulting equations are only simple if the "fluid" is microscopically isotropic; others will become apparent later.

We now collect together all of the "currents" mentioned above:

-Magnetization Current:

$$\vec{j}_M = \nabla \times \vec{M} = \nabla \times (n\vec{\mu}) = -\nabla \times \left(n\frac{\frac{1}{2}mv_\perp^2}{B}\frac{\vec{B}}{B} \right) \tag{35}$$

-"Curvature" or "Inertia" current:

$$\vec{j}_R = nq\frac{mv_\parallel^2}{qBR_c}\vec{bn} \tag{36}$$

-Grad-B current:

$$\vec{j}_{\nabla B} = nq\frac{mv_\perp^2}{2qB^2}\vec{b} \times \nabla B \tag{37}$$

-$E \times B$ current:

$$\vec{j}_{E\times B} = nq\frac{\vec{E} \times \vec{B}}{B^2} \tag{38}$$

Starting with (35), we can carry out the $\nabla \times (\cdots)$ operation:

$$\vec{j}_M = -\nabla\left(n\frac{1}{2}mv_\perp^2 \right) \times \frac{\vec{B}}{B^2} - n\frac{1}{2}mv_\perp^2\left(\frac{\nabla \times \vec{B}}{B^2} - \frac{2}{B^3}\nabla B \times \vec{B} \right) \tag{39}$$

Of the terms here, the first and last are clearly perpendicular to $\vec{B}$. Since the whole magnetization current arises from Larmor motion, which is perpendicular to $\vec{B}$, we should exclude

from the $\nabla \times \vec{B}$ in term in (38) the parallel part, and write instead $(\nabla \times \vec{B})_\perp$ in the middle term.

Add together all these currents:

$$\vec{j} = \vec{j}_M + \vec{j}_R + \vec{j}_{\nabla B} + \vec{j}_{E \times B} =$$
$$= -\nabla\left(\frac{1}{2}nmv_\perp^2\right) \times \frac{\vec{B}}{B^2} - \frac{1}{2}nmv_\perp^2\left(\frac{(\nabla \times \vec{B})_\perp}{B^2} + \frac{2b \times \nabla B}{B^2}\right) + n\frac{mv_\parallel^2}{BR_c}\vec{bn} \quad (40)$$
$$+ \frac{1}{2}nmv_\perp^2\frac{\vec{b} \times \nabla B}{B^2} + nq\frac{\vec{E} \times \vec{B}}{B^2}$$

The first thing we notice is that the Grad-B current cancels out with $1/2$ of a similar portion of the magnetization current. The remainder of this term can be manipulated using,

$$\frac{\vec{b} \times \nabla B}{B} = \frac{\vec{bn}}{R_c} - \frac{(\nabla \times \vec{B})_\perp}{B}$$

to yield,

$$\vec{j} = -\nabla\left(\frac{1}{2}nmv_\perp^2\right) \times \frac{\vec{B}}{B^2} - \frac{1}{2}nmv_\perp^2\left(\frac{(\nabla \times \vec{B})_\perp}{B^2} + \frac{\vec{bn}}{BR_c} - \frac{(\nabla \times \vec{B})_\perp}{B^2}\right) + \frac{nmv_\parallel^2}{BR_c}\vec{bn} + nq\frac{\vec{E} \times \vec{B}}{B^2}$$

and we now cancel out the bothersome $(\nabla \times \vec{B})_\perp$ terms. Grouping terms,

$$\boxed{\vec{j} = -\nabla\left(\frac{1}{2}nmv_\perp^2\right) \times \frac{\vec{B}}{B^2} + \frac{nm\left(v_\parallel^2 - \frac{1}{2}v_\perp^2\right)}{BR_c}\vec{bn} + nq\frac{\vec{E} \times \vec{B}}{B^2}} \quad (41)$$

To connect with the fluid viewpoint, we now recognize the existence of different values of $v_\parallel$ and $v_\perp$ within the one species of particle being considered, and perform a statistical average of (42) for all such particles. We use the definitions of press and temperature:

$$\left\langle \frac{1}{2}nmv_\perp^2 \right\rangle \equiv P_\perp \equiv nkT_\perp \quad (42)$$

$$\left\langle nmv_\parallel^2 \right\rangle \equiv P_\parallel \equiv nkT_\parallel \quad (43)$$

And obtain, for "the fluid",

$$\boxed{\vec{j} = -\nabla P_\perp \times \frac{\vec{B}}{B^2} + \frac{nk(T_\parallel - T_\perp)}{BR_c}\vec{bn} + nq\frac{\vec{E} \times \vec{B}}{B^2}} \quad (44)$$

The force density due to $\vec{B}$ on the fluid is,

$$\left\langle nq\vec{v} \times \vec{B} \right\rangle = \vec{j} \times \vec{B}$$

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We there form $\vec{j} \times \vec{B}$ from (44). For any vector $\vec{a}$,

$$\frac{\left(\vec{a} \times \vec{B}\right) \times \vec{B}}{B^2} = \vec{B}\frac{\vec{a} \cdot \vec{B}}{B^2} - \vec{a} = -\vec{a}_\perp$$

and also $(\vec{bn}) \times \vec{B} = \vec{n}B$.

Hence,

$$\vec{j} \times \vec{B} = +\nabla_\perp(P_\perp) + \frac{nk\left(T_\| - T_\perp\right)}{R_c}\vec{n} - nq\vec{E}_\perp$$

or finally,

$$\boxed{\nabla_\perp P_\perp + \frac{P_\| - P_\perp}{R_c}\vec{n} = nq\vec{E}_\perp + \vec{j} \times \vec{B}} \qquad (45)$$

The first term on the left is the opposite of familiar fluid pressure gradient force (perpendicular component). The two terms on the right are the electrostatic force density and the magnetic force density, respectively. The term,

$$\frac{P_\perp - P_\|}{R_c}\vec{n}$$

is unfamiliar in fluid terms, because one normally assumes <u>isotropy</u> in a fluid ($P_\perp = P_\|$), and this "<u>anisotropy force</u>" would be absent then.

In summary, for an <u>isotropic</u> fluid, the full curvature and gradient drifts are cancelled out by the $\vec{B} - field$ gradient terms in the magnetization current. This confirms the statement about how dangerous it may be to rely on one or other of the drifts in isolation.

For an example of this twist, consider a straight, long cable carrying a current I and embedded in a plasma.

$$\vec{B} = B(r)\vec{b}$$
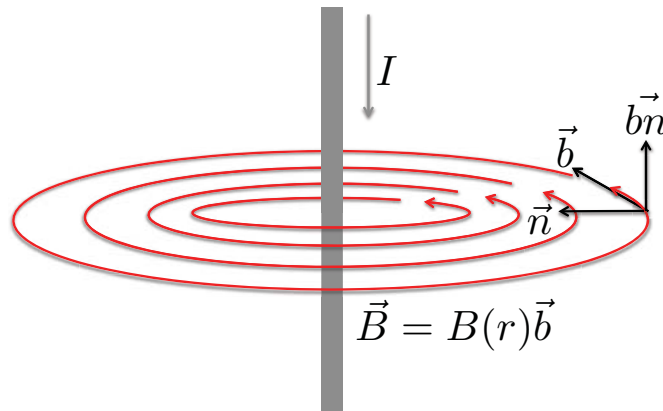
From Ampere's law, the $\vec{B}$ field set up by the current is,

$$B = \frac{\mu_0 I}{2\pi r}$$

and so it has both, curvature $(1/r)$ and a gradient,

$$\nabla B \;=\; \frac{\mu_0 I}{2\pi r^2}\vec{n} \;=\; \frac{B}{r}\vec{n}$$

According to (36) and (37), the drifts are,

$$\vec{j}_R \;=\; \frac{nmv_\parallel^2}{B_r}\vec{bn} \tag{46}$$

$$\vec{j}_{\nabla B} \;=\; \frac{nmv_\perp^2/2}{B^2}\frac{B}{r}\vec{b}\times\vec{n} \;=\; \frac{nmv_\perp^2/2}{Br}\vec{bn} \tag{47}$$

Both (46) and (47) are currents parallel to the wire, for any polarity, and one would be tempted to predict kind of "induced current" being "dragged along" by the main wire current. However, reference to (41) tells us that, if the plasma is uniform and isotropic, no such current exists.

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