Equilibrium in a Force Field

Let $Z(\vec{x}, \vec{w})$ be any <u>constant of the motion</u> for one of the particles (such as its total energy, or, in some cases its angular momentum or other properties). Since the Vlasov operator (left side of the Boltzmann equation) is simply the total rate of change of f, we can see that <u>any function $f(Z)$ will satisfy Vlasov's equation</u> (or Boltzmann's equation if there are no collisions).

In addition, we know that any distribution function of the form,

$$f = C(\vec{x})e^{-Dw^2 - \vec{G}\cdot\vec{w}}$$

with C, D and $\vec{G}$ arbitrary functions of position, has the property of making the Boltzmann collision operator identically zero. Thus, if the quantity,

$$-Dw^2 - \vec{G}\cdot\vec{w} + \ln C$$

happens to be a constant of the motion, then f, as defined, satisfies the full Boltzmann equation. An important example is furnished by the total energy of a charged particle in an electrostatic field:

$$Z(\vec{w}, \vec{x}) \; = \; \frac{1}{2}mw^2 \; + \; q\phi(\vec{x})$$

This is a constant of the motion (even if there is a static magnetic field as well, because $\vec{B}$ does not do work on a moving particle), and at the same time the function e^{-DZ} has the required velocity dependance to annul the collision operator. Thus, a solution to Boltzmann's equation in the presence of a potential $\phi(\vec{x})$ is,

$$f \; = \; Ce^{-D(\frac{1}{2}mw^2 + q\phi(\vec{x}))}$$

If this applies for all $\vec{w}$, then the <u>constants</u> C and D can be related to n and T in the usual way, by imposing,

$$n = \int f d^3w \qquad \text{and} \qquad n\frac{3}{2}kT = \int \frac{1}{2}mw^2 f d^3w$$

then we obtain,

$$\boxed{f \; = \; n_0 \left(\frac{m}{2\pi kT_0}\right)^{3/2} \exp\left(-\frac{\frac{1}{2}mw^2 + q\phi(\vec{x})}{kT_0}\right)}$$

Here, n_0 can be interpreted as the density where $\phi = 0$. At any other location, if this Maxwellian form holds for all $\vec{w}$, we can interpret the group $n_0 e^{-\frac{q\phi}{kT_0}}$ as the local density $n(\vec{x})$:

$$\boxed{n(\vec{x}) \; = \; n_0 \exp\left(-\frac{q\phi(\vec{x})}{kT_0}\right)} \qquad \text{Boltzmann's relationship}$$

Note: f still has to satisfy boundary conditions, so Boltzmann's relation may not apply in some conditions.

T_0 is the temperature where $\phi = 0$. It would also be the local T if the solution is full. Whether or not this solution applies in a specific case still depends on satisfying appropriate Boundary Conditions. In some cases it is found that these conditions are different for different $\vec{w}$, and discontinuities can arise in the form of the actual solution. Some examples will clarify this.

Example 1: Distribution function near an attracting wall. ($q\phi < 0$)
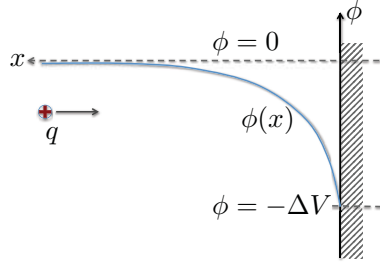
In the situation, we assume there is an equilibrated distribution for particles of charge q far from the wall, where the potential is $\phi_\infty = 0$. So, far form the wall,

$$f(\vec{w}, x \to \infty) = n_\infty \left(\frac{m}{2\pi k T_\infty} \right)^{3/2} e^{-\frac{mw^2}{2kT_\infty}}$$

The $\frac{1}{2} m w^2$ in the exponent is the full energy E of the particle, but at locations near the wall, the same particle has the same E, only it now satisfies,

$$E = \frac{1}{2} m w^2 + q\phi(x)$$

We can then say that for particles that come from $(+\infty)$, the distribution is,

$$f(\vec{w}, x) = n_\infty \left(\frac{m}{2\pi k T_\infty} \right)^{3/2} \exp\left(-\frac{\frac{1}{2} m w^2 + q\phi(x)}{kT_\infty} \right) \qquad (\phi \le 0)$$

But particles arriving from $(-\infty)$ have velocities limited by,

$$w_x \ge \sqrt{\frac{2q|\phi(x)|}{m}} \qquad \text{for any } w_y, w_z$$

and this is the only range for which the above distribution applies. If the wall absorbs all the arriving particles and returns none, then,
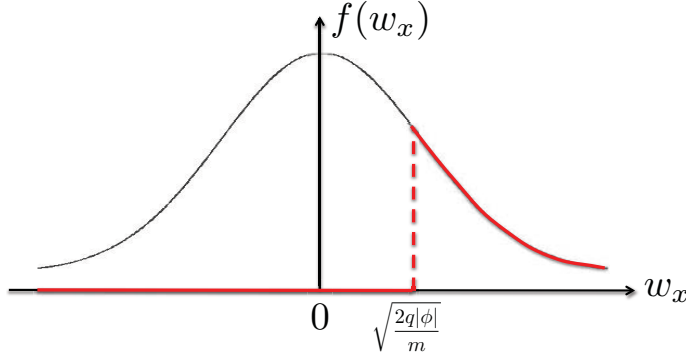
$$f(w_x < 0) = 0$$

and even stronger,

$$f\left(w_x < \sqrt{\frac{2q|\phi(x)|}{m}} \right) = 0$$

which is the limit corresponding to particles that start from $(-\infty)$ with nearly zero velocity towards the wall.

We can now sketch this solution, at least its w_x dependence (notice that the w_y, w_z dependencies are the usual, complete Maxwellian, from $-\infty$ to $+\infty$):

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Since a sizable part of the distribution is <u>not</u> Maxwellian, the relationship $n(x) = n_0 \exp\left(-\frac{q\phi}{kT_\infty}\right)$ does <u>not</u> apply in this case. Nevertheless, a direct integration can be performed.

There is an objection to this solution, however: as $x \to -\infty$, this tends to a <u>half-Maxwellian</u> ($w_x > 0$) and not to the assumed full equilibrium. Collisions and the finite dimensions of our "wall" (3D effects) need to be invoked to re-fill the $w_x < 0$ part of the distribution.

Example 2: Distribution function near a repelling wall

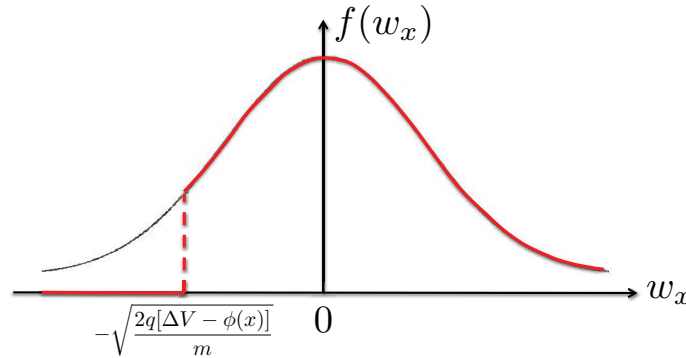Once again, a candidate distribution is:

$$f(\vec{w}, x) = n_\infty \left(\frac{m}{2\pi k T_\infty}\right)^{3/2} \exp\left(-\frac{\frac{1}{2}mw^2 + q\phi(x)}{kT_\infty}\right)$$
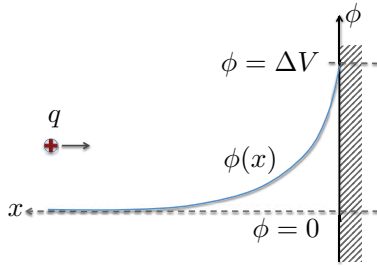
This time <u>any</u> positive w_x is allowed. Even particles with $w_x(-\infty) > 0$ can be stopped by the retarding potential.

However, for $w_x < 0$ the fastest particles present would be the ones that were turned back very near the wall:

$$|w_x| < \sqrt{\frac{2q(\Delta V - \phi(x))}{m}} \qquad (w_x < 0)$$

Notice that when $\Delta V \gg \frac{kT}{q}$ the excluded region corresponds to highly supra-thermal particles, which have low population, and the full range could be assumed with negligible error. In particular, then $n(x) \approx n_\infty e^{-\frac{q\phi(x)}{kT_\infty}}$ to a good approximation.

The generalization of these results is worth stating: The Boltzmann relation $n = n_\infty e^{-\frac{q\phi(x)}{kT_\infty}}$ is reliable when the particles are generally <u>contained</u> by the potentials, so that the net fluxes are small. It is <u>not</u> accurate when the potentials tend to extract particles and

create strong fluxes.

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