16.346 Astrodynamics
Fall 2008

Hyperbolic Orbits

$$x = a\sec\zeta \qquad y = b\tan\zeta \quad \Longleftrightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad r = a - ex \quad \Longrightarrow \quad \boxed{r = a(1 - e\sec\zeta)}$$

To understand the geometrical significance of ζ, write the equation of orbit as

$$r + re\cos f = a(1-e^2)$$
$$a(1 - e\sec\zeta) + re\cos f = a(1-e^2)$$
$$\underbrace{-a\sec\zeta}_{\text{positive}} + \underbrace{r\cos f}_{\text{negative}} = -ae = FC$$

and relate the terms of the last equation to the diagram

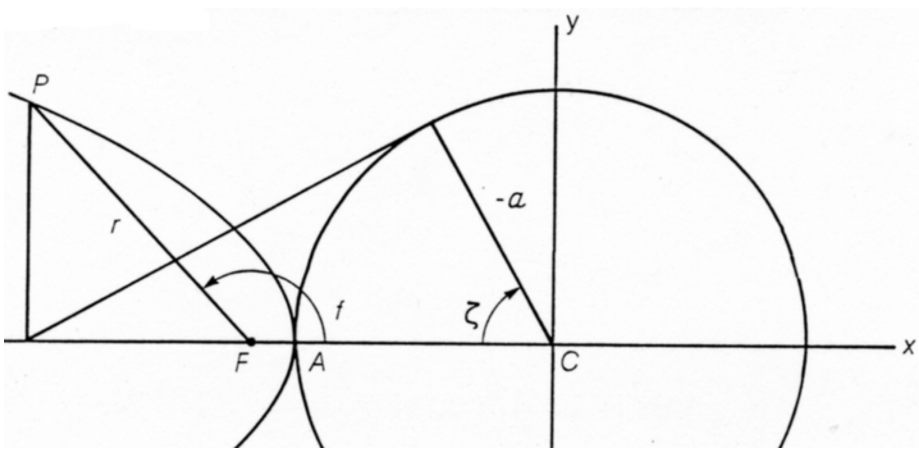

Fig. 4.12 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

Also

$$x = a\cosh H \qquad y = b\sinh H \quad \Longleftrightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad r = a - ex \quad \Longrightarrow \quad \boxed{r = a(1 - e\cosh H)}$$

Then $\qquad \tan\tfrac{1}{2}f = \sqrt{\dfrac{e+1}{e-1}}\tan\tfrac{1}{2}\zeta \qquad$ or $\qquad \boxed{\tan\tfrac{1}{2}f = \sqrt{\dfrac{e+1}{e-1}}\tanh\tfrac{1}{2}H}$

and the analogs of Kepler's equation are

$$N = e\tan\zeta - \log\tan(\tfrac{1}{2}\zeta + \tfrac{1}{4}\pi) \qquad \text{or} \qquad \boxed{N = e\sinh H - H}$$

where $\qquad \boxed{N = \sqrt{\dfrac{\mu}{(-a)^3}}\,(t-\tau)}$

Lagrange's Equations for Hyperbolic Orbits

For hyperbolic orbits, ψ and ϕ are defined as

$$\psi = \tfrac{1}{2}(H_2 - H_1) \qquad \cosh\phi = e\cosh\tfrac{1}{2}(H_1 + H_2)$$

and the basic equations are

$$\begin{aligned}
\sqrt{\mu}(t_2 - t_1) &= 2(-a)^{\frac{3}{2}}(\sinh\psi\cosh\phi - \psi) \\
r_1 + r_2 &= 2a(1 - \cosh\psi\cosh\phi) \\
c &= -2a\sinh\psi\sinh\phi \\
\sqrt{r_1 r_2}\cos\tfrac{1}{2}\theta &= a(\cosh\psi - \cosh\phi)
\end{aligned}$$

The Lagrange parameters are defined as for the ellipse. Then

$$\sqrt{\mu}(t_2 - t_1) = (-a)^{\frac{3}{2}}[(\sinh\alpha - \alpha) - (\sinh\beta - \beta)]$$

where

$$\sinh^2\tfrac{1}{2}\alpha = -\frac{s}{2a} \qquad \sinh^2\tfrac{1}{2}\beta = -\frac{s-c}{2a}$$

Hyperbolic Injection Velocity

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Recall to the velocity vector in terms of the semimajor axis :

$$\mathbf{v}_1 = \left(\sqrt{\frac{\mu}{2(s-c)} - \frac{\mu}{4a}} + \sqrt{\frac{\mu}{2s} - \frac{\mu}{4a}}\right)\mathbf{i}_c + \left(\sqrt{\frac{\mu}{2(s-c)} - \frac{\mu}{4a}} - \sqrt{\frac{\mu}{2s} - \frac{\mu}{4a}}\right)\mathbf{i}_{r_1}$$

Then

$$\lim_{\substack{\theta=\text{const.}\\ r_2\to\infty}} \mathbf{i}_c = \mathbf{i}_\infty \qquad \lim_{\substack{\theta=\text{const.}\\ r_2\to\infty}} \frac{1}{s} = 0 \qquad \lim_{\substack{\theta=\text{const.}\\ r_2\to\infty}} \frac{s}{r_2} = 1$$

Now

$$\frac{1+\cos\theta}{2} = \cos^2\tfrac{1}{2}\theta = \frac{s(s-c)}{r_1 r_2} \qquad \text{and} \qquad \lim_{\substack{\theta=\text{const.}\\ r_2\to\infty}} \frac{s(s-c)}{r_1 r_2} = \frac{1}{r_1}\times 1\times \lim_{\substack{\theta=\text{const.}\\ r_2\to\infty}} (s-c)$$

so that

$$\lim_{\substack{\theta=\text{const.}\\ r_2\to\infty}} (s-c) = \frac{r_1}{2}(1+\cos\theta)$$

Also, from the vis-viva integral: $v^2 = \mu\left(\dfrac{2}{r} - \dfrac{1}{a}\right) \implies \boxed{v_\infty^2 = \dfrac{\mu}{-a}}$

Therefore, in the limit: $\boxed{\mathbf{v}_1 = (D + \tfrac{1}{2}v_\infty)\,\mathbf{i}_\infty + (D - \tfrac{1}{2}v_\infty)\,\mathbf{i}_{r_1}}$

where $\boxed{D = \sqrt{\dfrac{v_\circ^2}{1+\cos\theta} + \dfrac{v_\infty^2}{4}}}$ and $\boxed{\cos\theta = \mathbf{i}_{r_1}\cdot\mathbf{i}_\infty}$

NOTE: $v_\circ$ is the circular speed at the pericenter radius r_1, i.e., $v_\circ^2 = \dfrac{\mu}{r_1}$.

Injection from Pericenter of a Hyperbolic Orbit

For injection from pericenter of the hyperbola

$$0 = \mathbf{i}_{r_1} \cdot \mathbf{v}_1 = (D + \tfrac{1}{2}v_\infty)\cos\theta + (D - \tfrac{1}{2}v_\infty)$$

so that

$$(1+\cos\theta)D = \tfrac{1}{2}v_\infty(1-\cos\theta)$$

Hence,

$$\boxed{\mathbf{v}_1 = \frac{v_\infty}{1+\cos\theta}(\mathbf{i}_\infty - \cos\theta\,\mathbf{i}_{r_1})}$$

To determine $\cos\theta$, first square both sides of the equation for D

$$(1+\cos\theta)^2D^2 = \tfrac{1}{4}v_\infty^2(1-\cos\theta)^2$$

Then, from the previous equation for D,

$$D = \sqrt{\frac{v_\circ^2}{1+\cos\theta} + \frac{v_\infty^2}{4}}$$

we also have

$$(1+\cos\theta)^2D^2 = v_\circ^2(1+\cos\theta) + \tfrac{1}{4}v_\infty^2(1+\cos\theta)^2$$

Therefore:

$$v_\circ^2(1+\cos\theta) + \tfrac{1}{4}v_\infty^2(1+\cos\theta)^2 = \tfrac{1}{4}v_\infty^2(1-\cos\theta)^2$$

Then $\boxed{\cos\theta = \cos(\tfrac{1}{2}\pi + \nu) = -\sin\nu}$ $\Longrightarrow$ $\boxed{\sin\nu = \dfrac{1}{1 + \dfrac{v_\infty^2}{v_\circ^2}}}$

where ν is the angle between the hyperbolic asymptote and the minor axis.

Out-of-Plane Injection from Pericenter of a Hyperbolic Orbit

The vector $\mathbf{v}_\infty = v_\infty\mathbf{i}_\infty$ is in the orbital transfer plane in which both P_1 and P_2 lie. The orientation of this plane can be specified by the vector $\mathbf{i}_N$ defined as $\mathbf{i}_N = \mathrm{Unit}(\mathbf{r}_1 \times \mathbf{v}_\infty)$. where $\mathbf{r}_1$ is the vector position of point P_1

Having determined $\cos\theta$ from

$$v_\circ^2(1+\cos\theta) + v_\infty^2\cos\theta = 0 \qquad \text{or} \qquad \cos\theta = -\frac{v_o^2}{v_o^2 + v_\infty^2}$$

and knowing $\mathbf{i}_\infty$, we can obtain the pericenter direction $\mathbf{i}_{r_m}$ using the vector rotation calculation developed in Lecture 12 on Page 2.

Rotate $\mathbf{i}_\infty$, *clockwise*, through the angle θ to obtain $\mathbf{i}_{r_m}$. Specifically:

$$\boxed{\mathbf{i}_{r_m} = \mathbf{i}_\infty - \sin\theta(\mathbf{i}_N \times \mathbf{i}_\infty) + (1-\cos\theta)\mathbf{i}_N \times (\mathbf{i}_N \times \mathbf{i}_\infty)}$$