16.346 Astrodynamics
Fall 2008

Equation of a parabolic orbit $e = 1$ #4.2

$$r = \frac{p}{1+\cos f} = \frac{p}{2\cos^2 \frac{1}{2} f} = \tfrac{1}{2} p \sec^2 \tfrac{1}{2} f \quad \Longrightarrow \quad \boxed{r = \tfrac{1}{2} p (1 + \tan^2 \tfrac{1}{2} f)}$$

Barker's Equation

Thomas Barker (1722–1809)

$$r^2\, df = \sqrt{\mu p}\, dt \quad \Longrightarrow \quad \boxed{\tan^3 \tfrac{1}{2} f + 3 \tan \tfrac{1}{2} f = 2B} \quad \text{where} \quad B = 3\sqrt{\frac{\mu}{p^3}}\,(t-\tau)$$

Solving Barker's Equation

- **Jerome Cardan's Method (From Tartaglia) 1545**

  Substitute $\tan \frac{1}{2} f = z - \dfrac{1}{z}$ to obtain $\left(z - \dfrac{1}{z}\right)^3 + 3\left(z - \dfrac{1}{z}\right) = 2B$

  or $z^6 - 2Bz^3 - 1 = 0$ from which $z = \left(B \pm \sqrt{1+B^2}\,\right)^{\frac{1}{3}}$ (Either sign gives the same solution.) Hence

$$\boxed{\tan \tfrac{1}{2} f = \left(B + \sqrt{1+B^2}\,\right)^{\frac{1}{3}} - \left(B + \sqrt{1+B^2}\,\right)^{-\frac{1}{3}}}$$

- **Karl Stumpff's Method (1959)**

$$\boxed{A = \left(B + \sqrt{1+B^2}\,\right)^{\frac{2}{3}} \qquad \tan \tfrac{1}{2} f = \frac{2AB}{1 + A + A^2}}$$

- **François Vièta's Method (1591)**

  Using $\tan^3 \frac{1}{2} f + 3 \tan \frac{1}{2} f = 2B$ write $\begin{cases} \tan \frac{1}{2} f = 2 \sinh \frac{1}{3} x \\ B = \sinh x \end{cases}$

  and the cubic equation becomes $4 \sinh^3 \frac{1}{3} x + 3 \sinh \frac{1}{3} x = \sinh x$ which is a standard identity for the hyperbolic sine. The solution is

$$x = \operatorname{arcsinh} B = \log(B + \sqrt{1+B^2}\,)$$

$$\boxed{\begin{aligned} x &= a\cos E \\ y &= b\sin E \end{aligned}} \qquad \Longleftrightarrow \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where E, called the "eccentric anomaly", was so-named by Kepler.

With the origin of coordinates at the center of the ellipse C, the equation of orbit is

$$r + ex = a$$

Therefore:

$$r + ea\cos E = a \qquad \Longrightarrow \qquad \boxed{r = a(1 - e\cos E)} \tag{4.28}$$

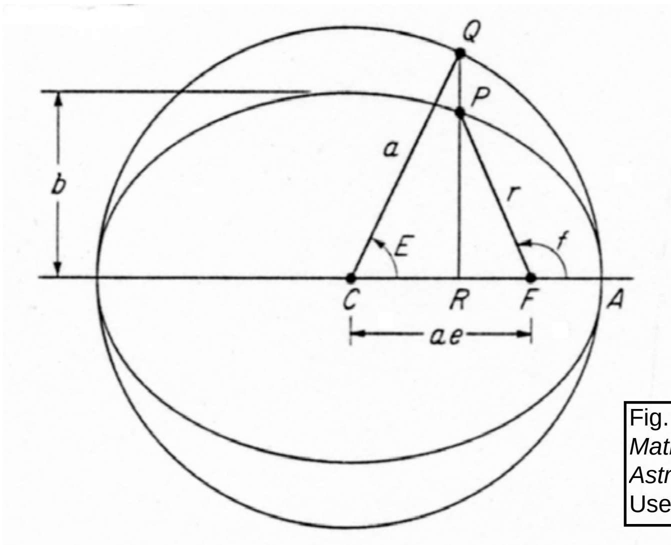

Fig. 4.9 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

The relation $y = b\sin E$ can be obtained as follows:

$$y^2 = PR^2 = PF^2 - RF^2 = a^2(1 - e\cos E)^2 - (ae - a\cos E)^2 = a^2(1 - e^2)\sin^2 E = b^2\sin^2 E$$

We have two expressions for the equation of orbit: one with the true anomaly f and the other with the eccentric anomaly E

$$r = \frac{a(1-e^2)}{1 + e\cos f} \qquad \text{and} \qquad r = a(1 - e\cos E)$$

from which we obtain relations between f and E

$$\cos f = \frac{\cos E - e}{1 - e\cos E} \qquad \text{or} \qquad \cos f = \frac{a}{r}(\cos E - e)$$

From these we derive

$$\underbrace{1 - \cos f}_{2\sin^2\frac{1}{2}f} = \frac{a}{r}(1+e)(\underbrace{1 - \cos E}_{2\sin^2\frac{1}{2}E}) \qquad \text{and} \qquad \underbrace{1 + \cos f}_{2\cos^2\frac{1}{2}f} = \frac{a}{r}(1-e)(\underbrace{1 + \cos E}_{2\cos^2\frac{1}{2}E})$$

Hence

$$\boxed{\tan\tfrac{1}{2}f = \sqrt{\frac{1+e}{1-e}}\tan\tfrac{1}{2}E} \tag{4.32}$$

Kepler's Equation

Take the differential of the identity relating $\tan \frac{1}{2} f$ and $\tan \frac{1}{2} E$:

$$\underbrace{\sec^2 \tfrac{1}{2} f \; df}_{=\frac{r}{a(1-e)} \sec^2 \frac{1}{2} E} = \sqrt{\frac{1+e}{1-e}} \sec^2 \tfrac{1}{2} E \, dE \quad \Longrightarrow \quad \boxed{r\, df = b\, dE} \tag{4.33}$$

Then $$\underbrace{r^2\, df = h\, dt}_{\text{Kepler's 2nd Law}} = br\, dE = ab(1 - e\cos E)\, dE \qquad \text{and} \qquad \frac{h}{ab} = \underbrace{\sqrt{\frac{\mu}{a^3}} = n}_{\text{Mean motion}}$$

Finally, integrate

$$n\, dt = (1 - e\cos E)\, dE$$
$$n(t - \tau) = E - e\sin E$$

where τ is the constant of integration and called the time of pericenter passage. Introducing the mean anomaly $M = n(t-\tau)$, we have the final form of Kepler's Equation

$$\boxed{M = E - e\sin E} \tag{4.34}$$

Solving Kepler's Equation Using a Cycloid – Isaac Newton See Prob 5–1

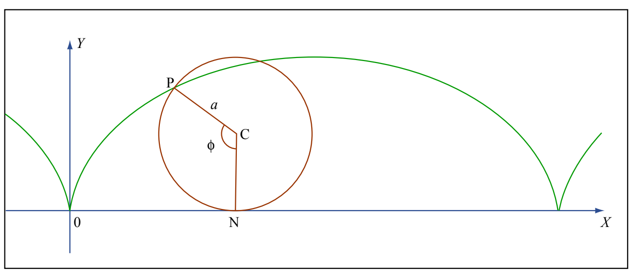

Figure by MIT OpenCourseWare.

$$x = a\phi - a\sin\phi$$
$$y = a - a\cos\phi$$

Solving Kepler's Equation by Successive Substitutions

$$\begin{aligned} E_0 &= 0 \\ E_1 &= M + e\sin E_0 \\ &\vdots \\ E_{k+1} &= M + e\sin E_k \quad \text{etc.} \end{aligned}$$

For the proof of convergence, we utilize the inequality See Pages 196–197

$$|E_{k+n} - E_k| \leq M \frac{e^k}{1-e}$$

which can be made arbitraily small (i.e., less than ϵ) by choosing

$$k > \log \frac{\epsilon(1-e)}{M} \Big/ \log e$$

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Carl Friedrich Gauss
1777–1855

The following is a translation from the original Latin into English of the first two sections of Gauss' book universally known as *Theoria Motus*. This was the major contribution of Gauss to the field of Celestial Mechanics published in 1809. The translation was motivated by an interest of the Navy Department and published in 1857 to meet the needs of the *American Ephemeris and Nautical Almanac* as well as the American Astronomers for whom the work was regarded as a standard and authority. In 1963 it was republished by Dover Publications, Inc. when a renewed interest in Celestial Mechanics was inspired by the requirements of the US space program.

Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections

Karl Friedrich Gauss

1.

In this work we shall consider the motions of the heavenly bodies so far only as they are controlled by the attractive force of the sun. All the secondary planets are therefore excluded from our plan, the perturbations which the primary planets exert upon each other are excluded, as is also all motion of rotation. We regard the moving bodies themselves as mathematical points, and we assume that all motions are performed in obedience to the following laws, which are to be received as the basis of all discussion in this work.

I. The motion of every heavenly body takes place in the same fixed plane in which the center of the sun is situated.

II. The path described by a body is a conic section having its focus in the center of the sun.

III. The motion in this path is such that the areas of the spaces described about the sun in different intervals of time are proportional to those intervals. Accordingly, if the times and spaces are expressed in numbers, any space whatever divided by the time in which it is described gives a constant quotient.*

IV. For different bodies moving about the sun, the squares of these quotients are in the compound ratio of the parameters of their orbits, and of the sum of the masses of the sun and the moving bodies.

Denoting, therefore, the parameter† of the orbit in which the body moves by p, the mass of this body by m (the mass of the sun being put $= 1$), the area it describes about the sun in the time t by $\frac{1}{2}g$, then‡ $\dfrac{g}{t\sqrt{p}\sqrt{1+m}}$ will be a constant§ for all heavenly

* Space/Time $= \frac{1}{2}h$.

† Gauss calls p the semi-parameter.

‡ In our notation $g = ht$ with $h = \sqrt{\mu p}$ so that $\dfrac{g}{t\sqrt{p}} = \sqrt{G(m_1+m_2)} = \sqrt{G(1+m)}$

§ $= \sqrt{G}$

bodies. Since then it is of no importance which body we use for determining this number, we will derive it from the motion of the earth, the mean distance of which from the sun we shall adopt for the unit of distance; the mean solar day will always be our unit of time. Denoting, moreover, by π the ratio of the circumference of the circle to the diameter, the area of the entire ellipse described by the earth will evidently be $\pi\sqrt{p}$,¶ which must therefore be put $= \frac{1}{2}g$,** if by t is understood the sidereal year; whence, our constant becomes†† $\dfrac{2\pi}{t\sqrt{1+m}}$. In order to ascertain the numerical value of this constant, hereafter to be denoted by k, we will put, according to the latest determination, the sidereal year or $t = 365.2563835$, the mass of the earth, or $m = \frac{1}{354710} = 0.0000028192$, whence results‡‡

$$\begin{aligned} \log 2\pi &= 0.7981798684 \\ \text{Compl.} \log t &= 7.4374021852 \\ \text{Compl.} \log \sqrt{1+m} &= 9.9999993878 \\ \hline \log k &= 8.2355814414 \\ k &= 0.01720209895. \end{aligned}$$

2.

The laws above stated differ from those discovered by our own Kepler in no other respect than this, that they are given in a form applicable to all kinds of conic sections, and that the action of the moving body on the sun, on which depends the factor $\sqrt{m_1 + m_2}$, is taken into account. If we regard these laws as phenomena derived from innumerable and indubitable observations, geometry allows what action ought in consequence to be exerted upon bodies moving about the sun, in order that these phenomena may be continually produced. In this way it is found that the action of the sun upon the bodies moving about it is exerted just as if an attractive force, the intensity of which is reciprocally proportional to the square of the distance, should urge the bodies towards the centre of the sun. If now, on the other hand, we set out with the assumption of such an attractive force, the phenomena are deduced from it as necessary consequences. It is sufficient here merely to have recited these laws, the connection of which with the principle of gravitation it will be the less necessary to dwell upon in this place, since several authors subsequently to the eminent Newton have treated this subject, and among them the illustrious La Place, in that most perfect work the Mécanique Céleste, in such a manner as to leave nothing further to be desired.

¶ Area $= \pi ab = \pi\sqrt{ap}$

** so that $g = 2\pi\sqrt{p}$

†† In our notation $\dfrac{4\pi^2 a^3}{P^2} = G(m_1 + m_2) = k^2(m_{\text{sun}} + m_{\text{earth}})$

‡‡ **The value of $k = \sqrt{G}$ obtained by Gauss as k = 0.01720209895 has been used over a long period in numerical investigations. Therefore, it is current practice to retain this value and adjust the unit of length in accordance with the more accurate values of the earth's mass m and the sidereal year t now available. The value of 1.0000010178 for the astronomical unit was the value determined in 1992.**