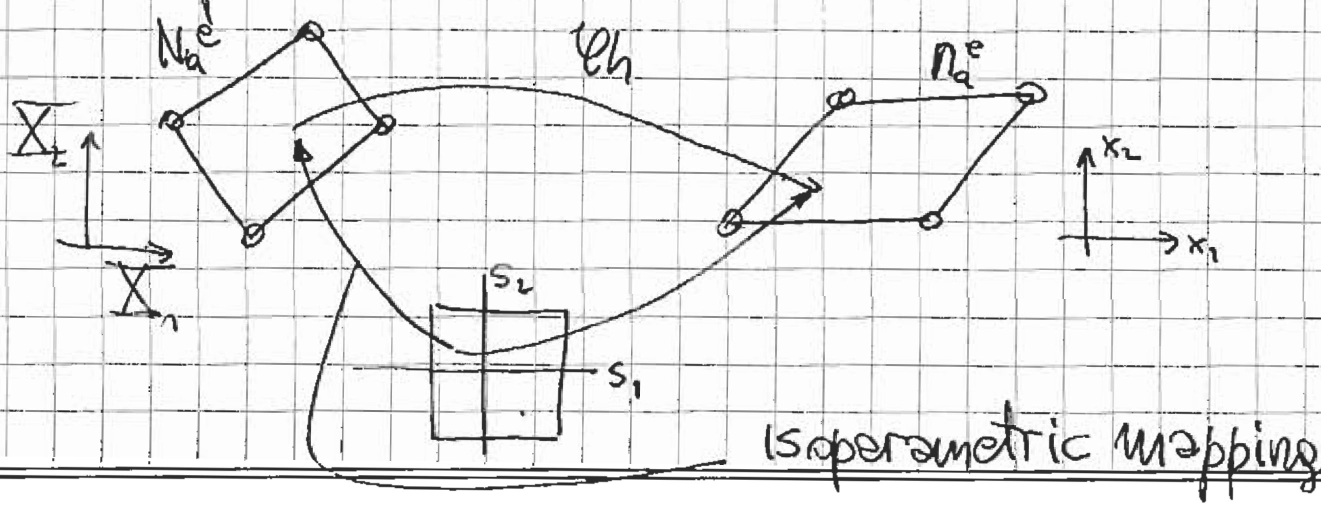
Introduce B-matrix (spatial)

$$\boxed{F^{int}_{ia} = \sum_{e=1}^{E} \int_{\Omega_t^e} B^{eT} \sigma(F_h)\, dV}$$

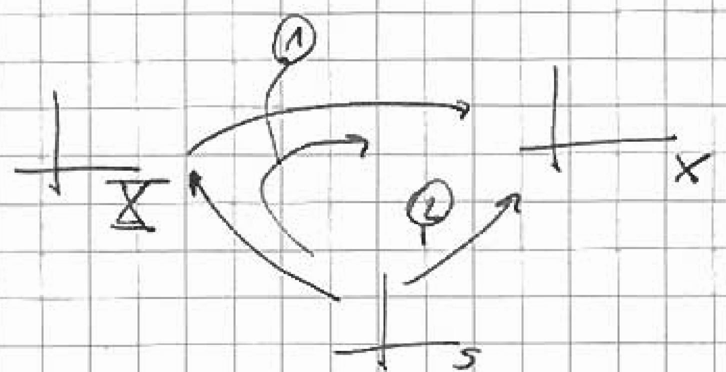
Isoparametric elements



$$\mathbb{X}_I = \sum_{a=1}^{n} \mathbb{X}_{Ia} \hat{N}_a(s)$$

$$x_i = \sum_{a=1}^{n} x_{ia} \hat{N}_a(s)$$

Commutative diagram! can go



ABSOLUTELY NO DIFFERENCE!!

Examples Two ways of computing $n^e_{a,j} \to$ shp

$N^e_{a,J} \to$ shp0

$\{x_{ia}\} \to xl$

$\{\mathbb{X}_{Ia}\} \to xl\emptyset$

1) call shape(, xl, ..., shp)

$\quad \hookrightarrow xl = xl\emptyset + ul$

② call shape(. — , xlφ, . — , shpφ)

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do i = 1, d
   do j = 1, n
      shp(i,j) = 0.d0
      do k = 1, d
         shp(i,j) += shpφ(k,j) * finv(k,i)
      end do
   end do
end do
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ISW = 6 → out-of-balance force

$$\Gamma_{ia}^{e} = \left(F_{ia}^{ext}\right)^{e} - \left(F_{ia}^{int}\right)^{e}$$

$$= \quad ① \equiv \int_{\Omega_0^e} \rho_0 B_i N_a^e \, dV_0 - \int_{\Omega_0^e} P_{iJ} N_{a,J} \, dV_0$$

$$= \quad ② \equiv \int_{\Omega_t^e} \rho \, b_i \, n_a^e \, dV - \int_{\Omega_t^e} B^{e^T} \sigma \, dV_t$$

can use either one

$$① = \sum_{p=1}^{Q} \rho_0 \, B_i \, N_a^e(\xi_p) \, w_p^e$$

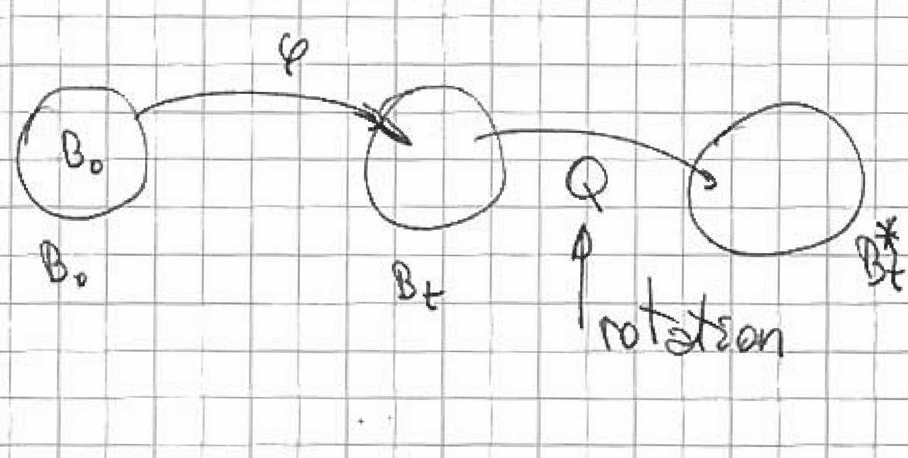
$$② = \sum_{p=1}^{Q} \rho \, b_i \, n_a^e(\xi_p) \, J_{(\xi_p)} w_p^e$$

Tangent stiffness

$$K_{iakb} = \sum_{e=1}^{E} \int_{\Omega_0^e} C_{iJkL} \, N^e_{a,J} \, N^e_{b,L} \, dV_0$$

$$\boxed{C_{iJkL} = \frac{\partial P_{iJ}}{\partial F_{kL}}} \qquad P_{iJ} = \frac{\partial W}{\partial F_{iJ}}(F)$$

Material frame indifference



$$W(QF) = W(F)$$
$$\forall Q \in SO(3)$$

$$Q^T Q = I \quad , \quad \det Q = 1$$

<u>Consequence</u> : $\quad W = W(C) \qquad C = F^T F$

$$P_{iJ} = \frac{\partial W}{\partial F_{iJ}}(C) = \frac{\partial W}{\partial C_{KL}} \frac{\partial C_{KL}}{\partial F_{iJ}}$$

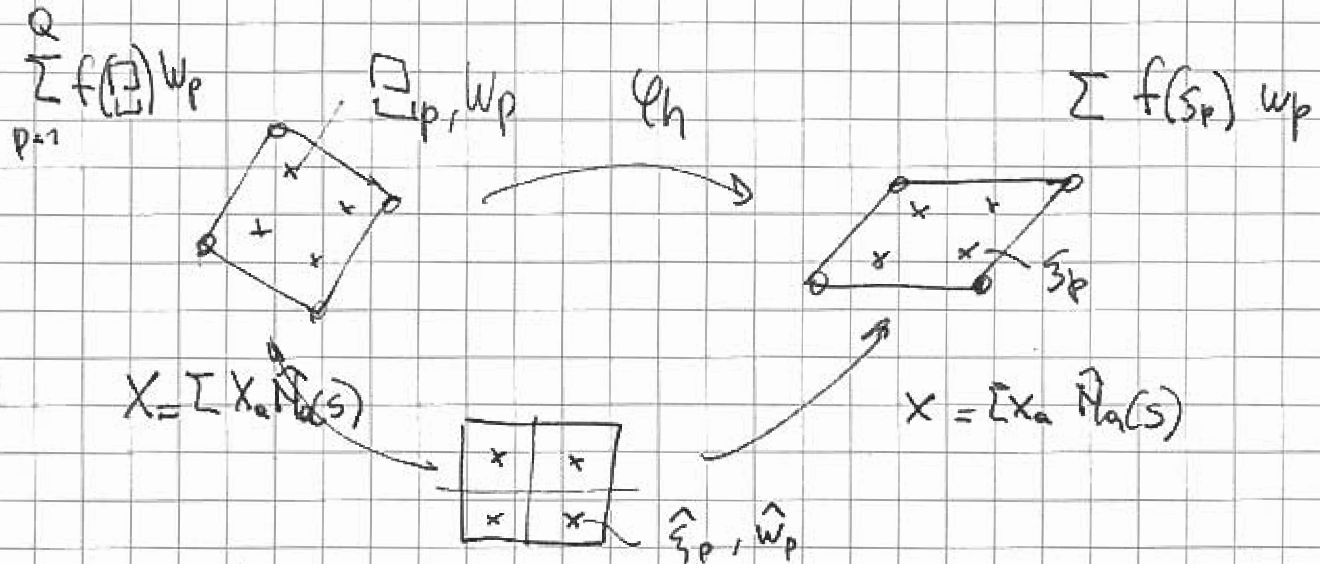
$$\frac{\partial C_{KL}}{\partial F_{iJ}} = \frac{\partial}{\partial F_{iJ}}\left(F_{nK} F_{nL}\right) = \delta_{in}\delta_{JK} F_{nL} + F_{nK}\delta_{in}\delta_{JL}$$

$$\boxed{P_{iJ} = 2\frac{\partial W}{\partial C_{IJ}}(C) \; F_{iI}}$$

$$\boxed{P_{iJ} = S_{IJ} F_{iI}} \longrightarrow$$

$$\boxed{S_{IJ} = 2\frac{\partial W}{\partial C_{IJ}}(C)} \iff \boxed{\dot{u} = S_{IJ}\frac{1}{2}\dot{C}_{IJ}}$$

Form of $C_{iJkL} = \frac{\partial P_{iJ}}{\partial F_{kL}}$ for materials satisfying material frame indifference.



$$\boxed{W_p = \hat{w}_p \underbrace{\det\left(\frac{\partial X}{\partial s}\right)}_{\text{xsj}\phi}}$$

$$w_p = \hat{w}_p \det\left(\frac{\partial x}{\partial s}\right)$$

$$w_p = W_p J$$

Calculation of tangent stiffness (cont'd)

$$C_{iJkL} = \frac{\partial P_{iJ}}{\partial F_{kL}} = \frac{\partial^2 W(F)}{\partial F_{iJ}\, \partial F_{kL}}$$

Material frame indifference $\quad W(F) = W(QF),\ Q \in SO(3)$

$\Rightarrow W(C)\ ,\quad C = F^T F \quad \rightarrow$

$$\boxed{S_{IJ} = \frac{2\,\partial W}{\partial C_{IJ}}(C)}$$

$$\boxed{P_{iJ} = S_{IJ} F_{iI} = 2\, F_{iI} \frac{\partial W}{\partial C_{IJ}}}$$

$$C_{iJkL} = \frac{\partial}{\partial F_{kL}} \left(2\, F_{iI} \frac{\partial W}{\partial C_{IJ}} \right)$$

$$= \underbrace{\delta_{ik}\, \delta_{IL}\, S_{IJ}}_{\delta_{ik}\, S_{JL}} + 2\, F_{iI} \frac{\partial^2 W}{\partial C_{IJ}\, \partial C_{MN}} \frac{\partial C_{MN}}{\partial F_{kL}}$$

$$\frac{\partial C_{MN}}{\partial F_{kL}} = \frac{\partial}{\partial F_{kL}} \left(F_{pM} F_{pN} \right) = \delta_{pk} \overset{\delta_{ML}}{} F_{pN} + F_{pM}\, \delta_{pk}\, \delta_{NL}$$

$$\Rightarrow C_{iJkL} = \delta_{ik}\, S_{JL} + 2\, F_{iI} \frac{\partial^2 W}{\partial C_{IJ}\, \partial C_{KL}} \left(\cdot \qquad \right)$$

$$C_{iJkL} = \delta_{ik} S_{JL} + 4 \frac{\partial^2 W(C)}{\partial C_{IJ} \partial C_{KL}} F_{iI} F_{kK}$$

Let $$\boxed{4 \frac{\partial^2 W(C)}{\partial C_{IJ} \partial C_{KL}} = C_{IJKL} \equiv \text{material moduli}}$$

Lagrangian moduli

$$\boxed{C_{iJkL} = \delta_{ik} S_{JL} + C_{IJKL} F_{iI} F_{kK}}$$

geometric stiffness

material stiffness

Tangent stiffness

$$K_{iakb} = \sum_{e=1}^{E} \int_{\Omega_0^e} C_{iJkL} N^e_{a,J} N^e_{b,L} \, dV_0$$

$$K_{iakb} = \sum_{e=1}^{E} \int_{\Omega_0^e} \left[\delta_{ik} S_{JL} + C_{IJKL} F_{iI} F_{kK} \right] N^e_{a,J} N^e_{b,L} \, dV_0$$

$$= \sum_{e=1}^{E} \int_{\Omega_t^e} \left[\delta_{ik} S_{JL} + C_{IJKL} F_{iI} F_{kK} \right] n^e_{a,j} n^e_{b,\ell} F_{jJ} F_{\ell L} J^{-1} dV$$

Let $\boxed{\text{spatial moduli}}$ $\boxed{c_{ijk\ell} = J^{-1} F_{iI} F_{jJ} F_{kK} F_{\ell L} C_{IJKL}}$

$\boxed{\sigma_{ij} = J^{-1} F_{jJ} F_{\ell L} S_{JL}}$

$$\rightarrow \boxed{K_{iakb} = \sum_{e=1}^{E} \int_{\Omega_t^e} \left[\delta_{ik} \sigma_{j\ell} + c_{ijk\ell} \right] n^e_{a,j} n^e_{b,\ell} \, dV_t}$$

Note $c_{ijk\ell}$ has major and minor symmetries
C_{IJKL}

However $\underline{\delta_{ik} \sigma_{j\ell}}$ has only the major symmetries

$$\Longrightarrow K_{iakb} = \sum_{e=1}^{E} \int_{\Omega_t^e} \left[\delta_{ik} \sigma_{jl} \, n^e_{a,j} \, n^e_{b,l} + \left(B^{e^T} C B^e \right)_{iakb} \right] dV$$

~~The~~ linearizing about non-stress-free configuration keeps the first term

LE: linearized version of the above

+ initial configuration is stress-free.

Remark: c_{ijkl} usually ~~are~~ have a simple form ~~in~~ than C_{iJkL} or C_{IJKL}.

Examples: $W(C)$ so far only restriction material frame indiference.