164B

# Examples: • Trapezoidal rule

$$P = A + \alpha \Delta t B$$
$$Q = A - (1-\alpha)\Delta t B$$

$P(0) = A \;,\; Q(0) = A \quad \checkmark \Rightarrow F(0) = I$

$P'(0) = \alpha B \;,\; Q'(0) = -(1-\alpha) B$

$$-A^{-1}\alpha B + A^{-1}(\alpha - 1) B = -A^{-1} B \quad \forall \alpha$$

$\Longrightarrow$ Trapezoidal rule is consistent for all $\alpha$

• Newmark's algorithm

$$P(0) = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \;,\; Q(0) = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$$

$$P'(0) = \begin{pmatrix} 0 & 0 \\ \gamma K & \gamma C \end{pmatrix} \;,\; Q'(0) = \begin{pmatrix} 0 & M \\ -(1-\gamma)K & -(1-\gamma)C \end{pmatrix}$$

$$-\begin{pmatrix} M^{-1} & 0 \\ 0 & M^{-1} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \gamma K & \gamma C \end{pmatrix} + \begin{pmatrix} M^{-1} & 0 \\ 0 & M^{-1} \end{pmatrix}\begin{pmatrix} 0 & M \\ -(1-\gamma)K & -(1-\gamma)C \end{pmatrix}$$

$$= \begin{pmatrix} M^{-1} & 0 \\ 0 & M^{-1} \end{pmatrix}\begin{pmatrix} 0 & M \\ \cancel{+\gamma K} - K \cancel{+\gamma K} & \cancel{+\gamma C} - C \cancel{+\gamma C} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix} = -\begin{pmatrix} 0 & I \\ M^{-1}K & M^{-1}C \end{pmatrix} \checkmark$$

$$A = \begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -K \\ K & C \end{pmatrix}$$

$$-A^{-1}B = -\begin{pmatrix} K^{-1} & 0 \\ 0 & M^{-1} \end{pmatrix}\begin{pmatrix} 0 & -K \\ K & C \end{pmatrix} = -\begin{pmatrix} 0 & I \\ M^{-1}K & M^{-1}C \end{pmatrix} \checkmark$$

$\forall \beta, \gamma$

$\Longrightarrow$ $\boxed{\text{Newmark is consistent } \forall \beta, \gamma}$

<u>Second order accuracy</u>: In addition to consistency require:

$$\left.\frac{d^2 F(\Delta t)}{dt^2}\right|_{\Delta t = 0} = \ddot{y}$$

$$\ddot{y}: \quad A\dot{y} + By = 0 \rightarrow \dot{y} = -A^{-1}By$$

$$\ddot{y} = -A^{-1}B\dot{y} = A^{-1}BA^{-1}By$$

Definition: $F(\Delta t)$ second order accurate if

$$\boxed{\left.\frac{d^2}{d\Delta t^2} F(\Delta t)\right|_{\Delta t = 0} = A^{-1}BA^{-1}B}$$

Express in terms of $P, Q$

$$PF = Q \rightarrow P'F + PF' = Q'$$

$$\rightarrow F' = P^{-1}(Q' - P'F)$$

$$P''F + P'F' + P'F' + PF'' = Q''$$

$$F'' = P^{-1}(Q'' - P''F - 2P'F')$$

$F''(0)$: $F(0) = I$, $F'(0) = -A^{-1}B$ (algorithm consistent)

$$F''(0) = P^{-1}(0)\left[Q''(0) - P''(0)I + 2P'(0)A^{-1}B\right]$$

2nd-order: $Q''(0) - P''(0) + 2P'(0)A^{-1}B = P(0)A^{-1}BA^{-1}B$

$$Q''(0) - P''(0) + \left(2P'(0) - P(0)A^{-1}B\right)A^{-1}B = 0$$

$$Q''(0) - P''(0) + [2P'(0) + P(0) \underbrace{(-A^{-1}B)}_{-P^{-1}(0)P'(0)+Q^{-1}(0)Q'(0)}] A^{-1}B = 0$$

$$Q''(0) - P''(0) + [2P'(0) - \underbrace{P(0)P^{-1}(0)}_{I} P'(0) + \underbrace{P(0)Q^{-1}(0)}_{I} Q'(0)$$

$$] A^{-1}B = 0 \qquad P(0) = Q(0)$$

$$\boxed{Q''(0) - P''(0) + [P'(0) + Q'(0)] A^{-1}B = 0}$$

$$\Longleftrightarrow \boxed{\frac{d^2}{d\Delta t^2} F(\Delta t)\Big|_{\Delta t=0} = A^{-1}BA^{-1}B} \Longleftrightarrow \text{"}F(\Delta t)\text{" second order accurate}$$

## Examples

- Trapezoidal rule: $P = A + \alpha \Delta t B$
  $Q = A - (1-\alpha)\Delta t B$

$$P'(0) = \alpha B \quad , \quad P''(\Delta t) = 0$$

$$Q'(0) = (\alpha - 1)B \quad , \quad Q''(\Delta t) = 0$$

2nd-order? $\quad 0 - 0 + [\alpha B + (\alpha - 1)B] A^{-1}B = 0$

$$(2\alpha - 1) BA^{-1}B = 0$$

Want rule 2nd order accurate independent of IVP $\Longrightarrow$ $(2\alpha - 1) = 0$

Trapezoidal rule is second order accurate for: $\boxed{\alpha = \frac{1}{2}}$ irrespective of the specific initial value problem

- Newmark's algorithm

$$P = \begin{pmatrix} M + \beta \Delta t^2 K & \beta \Delta t^2 C \\ \gamma \Delta t K & M + \gamma \Delta t C \end{pmatrix},$$

$$P' = \begin{pmatrix} 2\beta \Delta t K & 2\beta \Delta t C \\ \gamma K & \gamma C \end{pmatrix}, \quad P'(0) = \begin{pmatrix} 0 & 0 \\ \gamma K & \gamma C \end{pmatrix}$$

$$P'' = \begin{pmatrix} 2\beta K & 2\beta C \\ 0 & 0 \end{pmatrix}, \quad P''(0) = \begin{pmatrix} 2\beta K & 2\beta C \\ 0 & 0 \end{pmatrix}$$

$$Q = \begin{pmatrix} M - (1/2 - \beta) \Delta t^2 K & \Delta t M - (1/2 - \beta) \Delta t^2 C \\ -\Delta t (1-\gamma) K & M - \Delta t (1-\gamma) C \end{pmatrix}$$

$$Q' = \begin{pmatrix} (2\beta - 1) \Delta t K & M + (2\beta - 1) \Delta t C \\ (\gamma - 1) K & (\gamma - 1) C \end{pmatrix}, \quad Q'(0) = \begin{pmatrix} 0 & M \\ (\gamma - 1) K & (\gamma - 1) C \end{pmatrix}$$

$$Q'' = \begin{pmatrix} (2\beta-1)K & (2\beta-1)C \\ 0 & 0 \end{pmatrix} = Q''(0)$$

2nd order?

$$\begin{pmatrix} (2\beta-1)K & (2\beta-1)C \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 2\beta K & 2\beta C \\ 0 & 0 \end{pmatrix} +$$

$$+ \left[ \begin{pmatrix} 0 & 0 \\ \gamma K & \gamma C \end{pmatrix} + \begin{pmatrix} 0 & M \\ (\gamma-1)K & (\gamma-1)C \end{pmatrix} \right] \underbrace{A^{-1}B}_{\begin{pmatrix} 0 & I \\ M^{-1}K & M^{-1}C \end{pmatrix}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -K & -C \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & M \\ (2\gamma-1)K & (2\gamma-1)C \end{pmatrix} \begin{pmatrix} 0 & I \\ M^{-1}K & M^{-1}C \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -K & -C \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} K & C \\ (2\gamma-1)CM^{-1}K & (2\gamma-1)(K+CM^{-1}C) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ (2\gamma-1)CM^{-1}K & (2\gamma-1)(K+CM^{-1}C) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Longleftrightarrow \boxed{\gamma = \frac{1}{2}} \Longleftrightarrow$$ Newmark's algorithm is second order

## Stability

The following simple example shows that consistency alone is not enough for convergence

Trapezoidal rule, scalar problem:

$$\dot{y} + \lambda y = 0 \;, \; y(0) = y_0$$

$$\frac{y_{n+1} - y_n}{\Delta t} + \lambda\left[(1-\alpha) y_n + \alpha\, y_{n+1}\right] = 0$$

$$y_{n+1} = \underbrace{\frac{1 - \Delta t\,\lambda(1-\alpha)}{1 + \Delta t\,\lambda\,\alpha}}_{F(\Delta t)}\, y_n$$

Exact solution: $y(t) = e^{-\lambda t} y_0$

$t \to \infty \Rightarrow y \to 0$

Numerical:

$y_1 = F(\Delta t)\, y_0 \;, \; y_2 = F(\Delta t)\, y_1 = F^2(\Delta t)\, y_0 \;, \cdots y_n = F^n(\Delta t)\, y_0$

what happens when $t = n\,\Delta t \to \infty$, i.e. $n \to \infty$?

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} F^n(\Delta t)\, y_0 = 0 \iff$$

$$\underline{|F(\Delta t)| < 1}$$

$$\iff -1 < \frac{1-\Delta t\lambda(1-\alpha)}{1+\Delta t\lambda\alpha} < 1$$

$$-1-\Delta t\lambda\alpha < 1-\Delta t\lambda(1-\alpha) < 1+\Delta t\lambda\alpha$$

$$(\Delta t, \lambda, \alpha) \geq 0$$

- $-2 < \Delta t\lambda[\alpha-(1-\alpha)]$

$$\underbrace{-2 < \Delta t\lambda(2\alpha-1)} \rightarrow 2 > \Delta t\lambda(1-2\alpha)$$

most insightful

$$\boxed{\Delta t < \frac{2}{\lambda(1-2\alpha)}}$$

- $(\alpha-1)\cancel{\Delta t}\cancel{\lambda} < \cancel{\Delta t}\cancel{\lambda}\alpha \rightarrow -1 < 1$ always true

$\underline{\alpha > 1/2}$: $-2 < \Delta t\lambda b$, $b > 0$

$\Delta t > 0$ $\lambda > 0$ $\rightarrow$ always $\underline{\text{stable}}$

$\underline{\alpha = 1/2}$ $-2 < 0 \rightarrow \underline{\text{stable}}$

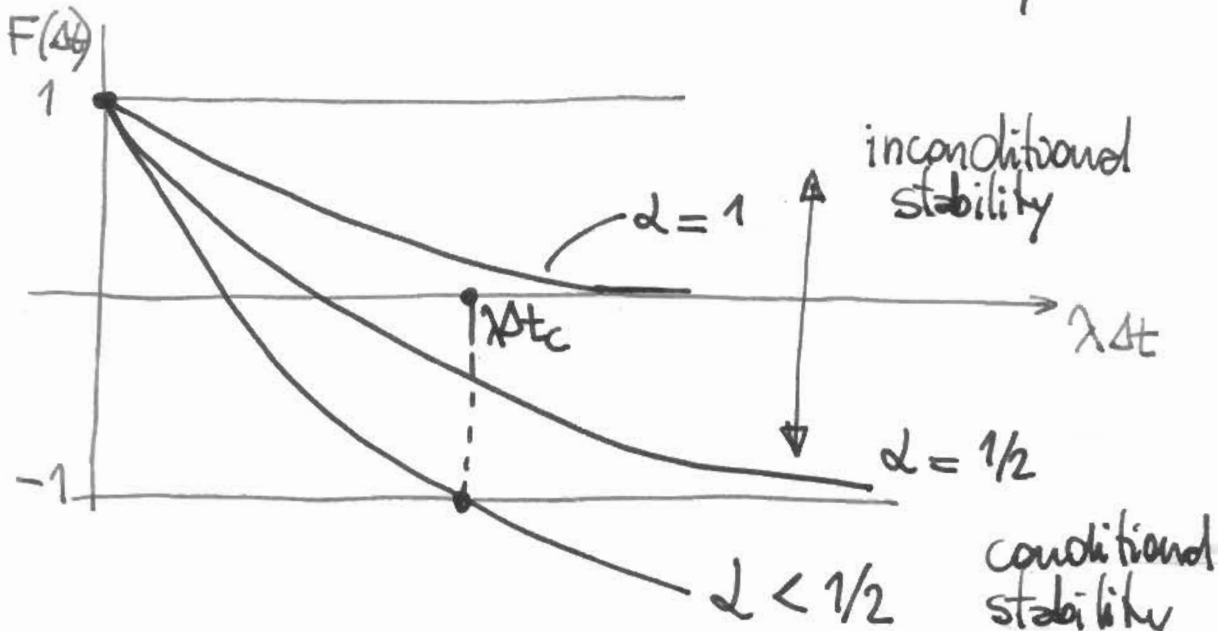
$\underline{\alpha < 1/2}$ $\boxed{\Delta t_c < \frac{2}{\lambda(1-2\alpha)}}$ conditionally stable

Conclusion: For $\alpha \geq 1/2$ $\Delta t_c := \infty \Longrightarrow \Delta t < \Delta t_c$

stability places no restriction on time step.

Time-step decided on the grounds of accuracy.

$\alpha < 1/2 \Rightarrow \Delta t < \Delta t_c$ to ensure stability.




Q: What happens if $|F(\Delta t)| > 1$, i.e. $\Delta t > \Delta t_c$?

A: $y_n$ diverges exponentially

Resolvent formula for exponential:

$$\lim_{n \to \infty} \left(1 + \frac{t}{n}\right)^{-n} = e^{-t}$$

Try to write algorithm this way:

$$y_n = F^n y_0 = \left(\frac{1}{F}\right)^{-n} y_0 = \Big[1 - \underbrace{\left(1 - \frac{1}{F}\right)}_{\epsilon}\Big]^{-n} y_0$$

$F > 1 \Leftrightarrow \epsilon > 0$

$$= (1-\epsilon)^{-n} y_0$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} (1-\epsilon)^{-n} y_0 = e^{\epsilon n} y_0$$

$$\rightarrow \boxed{y_n = e^{\epsilon t/\Delta t} y_0} \quad , \epsilon > 0$$

$\Rightarrow$ exponential divergence