Connection between Newmark algorithm and multistep methods.

① $Ma + f^{int}(x) = f^{ext}$

②$_n$ $x_{n+1} = x_n + \Delta t\, v_n + \Delta t^2 \left[\left(\tfrac{1}{2} - \beta\right) a_n + \beta\, a_{n+1}\right]$

③ $v_{n+1} = v_n + \Delta t \left[(1-\gamma)\, a_n + \gamma\, a_{n+1}\right]$

Write as a difference equation for "x_n".

Multiply $M \times$ ②:

$$0 = M\left(\frac{x_n + \Delta t\, v_n - x_{n+1}}{\Delta t^2}\right) + \left(\frac{1}{2} - \beta\right)\left(f_n^{ext} - f_n^{int}\right) + \beta\left(f_{n+1}^{ext} - f_{n+1}^{int}\right)$$

②$_{n-1}$ $x_n = x_{n-1} + \Delta t\, v_{n-1} + \Delta t^2 \left[\left(\tfrac{1}{2} - \beta\right) a_{n-1} + \beta\, a_n\right]$

$$0 = M\left(\frac{x_{n-1} + \Delta t\, v_{n-1} - x_n}{\Delta t^2}\right) + \left(\frac{1}{2} - \beta\right)\left(f_{n-1}^{ext} - f_{n-1}^{int}\right) + \beta\left(f_n^{ext} - f_n^{int}\right)$$

subtract

$$0 = M\,\frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta t^2} + \frac{M}{\Delta t}\left(v_{n-1} - v_n\right) +$$

$$\left(\beta - \frac{1}{2} + \beta\right)\left(f_n^{ext} - f_n^{int}\right) + \left(\frac{1}{2} - \beta\right)\left(f_{n-1}^{ext} - f_{n-1}^{int}\right) - \beta\left(f_{n+1}^{ext} - f_{n+1}^{int}\right)$$

$$v_n - v_{n-1} = \Delta t\left[(1-\gamma)\,a_{n-1} + \gamma\, a_n\right] \quad \text{from } ③$$

$$\frac{M}{\Delta t}\left(v_n - v_{n-1}\right) = (1-\gamma)\,M a_{n-1} + \gamma\, M a_n$$

$$= (1-\gamma)\left(f_{n-1}^{ext} - f_{n-1}^{int}\right) + \gamma\left(f_n^{ext} - f_n^{int}\right)$$

- Replacing in the previous equation:

$$0 = M\,\frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta t^2} - (1-\gamma)\left(f_{n-1}^{ext} - f_{n-1}^{int}\right) - \gamma\left(f_n^{ext} - f_n^{int}\right)$$

$$+ \left(2\beta - \frac{1}{2}\right)\left(f_n^{ext} - f_n^{int}\right) + \left(\frac{1}{2} - \beta\right)\left(f_{n-1}^{ext} - f_{n-1}^{int}\right) - \beta\left(f_{n+1}^{ext} - f_{n+1}^{int}\right)$$

$$0 = M\,\frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta t^2} + \left(\frac{1}{2} - \beta - 1 + \gamma\right)\left(f_{n-1}^{ext} - f_{n-1}^{int}\right) +$$

$$+ \left(2\beta - \frac{1}{2} - \gamma\right)\left(f_n^{ext} - f_n^{int}\right)$$

$$+(-\beta)\left(f^{ext}_{n+1}-f^{int}_{n+1}\right)$$

$$\boxed{0 = M\frac{x_{n+1}-2x_n+x_{n-1}}{\Delta t^2} + \alpha_{-1}\left(f^{ext}-f^{int}\right)_{n-1} + \alpha_0\left(f^{ext}-f^{int}\right)_n + \alpha_1\left(f^{ext}-f^{int}\right)_{n+1}}$$

$$\boxed{\alpha_{-1} = \gamma-\beta-\frac{1}{2}}$$

$$\boxed{\alpha_0 = 2\beta-\gamma-\frac{1}{2}}$$

$$\boxed{\alpha_1 = -\beta}$$

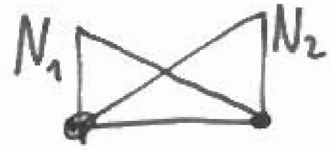
NEWMARK ALGORITHM IN MULTISTEP FORM

Mass lumping (similarly capacity lumping)

Consistent mass

$$M_{iakb} = \sum_e \int_{\Omega^e} \rho_0 \, \delta_{ik} \, N_a^e N_b^e \, dV$$

not diagonal in general:



$$\int_{\Omega^e} N_1 N_2 \, dV \neq 0$$

(1) Nodal quadrature

$$M_{iakb} = \sum_e \sum_q^Q w_q \, \rho_0 \, \delta_{ik} \, N_a^e(\xi_q) \, N_b^e(\xi_q)$$

if ξ_p's coincide with nodal points (Lobatto rules): $\int_{\Omega^e} N_a^e N_b^e \, dV = 0$, $a \neq b$

$\longrightarrow$ diagonal M

$Q=2$: $\displaystyle\int_{-1}^{1} f(x)\,dx = f(-1) + f(1)$ (trapezoidal rule)

$Q=3$: $\displaystyle = \frac{1}{3}\left(f(-1) + 4f(0) + f(1)\right)$ (Simpson's rule)

$$M_{ab}^{e} = \int_0^{\Delta x} \rho\, N_a^e N_b^e \, dx$$

$$M_{ab}^{e} = \rho\, \Delta x \int_0^1 N_a^e N_b^e \, d\xi \qquad \xi = \frac{x}{\Delta x}$$

$$\int_0^1 \left(N_i^e\right)^2 d\xi = \frac{1}{3} \quad , \quad \int_0^1 N_1 N_2 \, d\xi = \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}$$

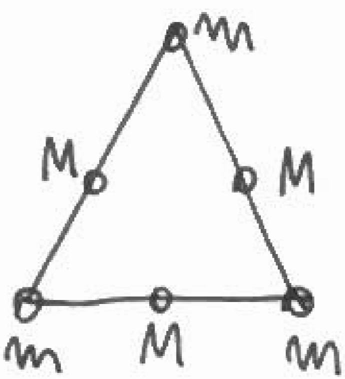
$$\boxed{M = \rho\, \Delta x \begin{pmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{pmatrix}}$$

$$\boxed{M^{\text{lumped}} = \rho\, \Delta x \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}}$$

- <u>3-node triangle:</u>

$$M^{\text{lumped}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rho_0 \frac{A^e}{3}$$

- <u>6-node triangle:</u>



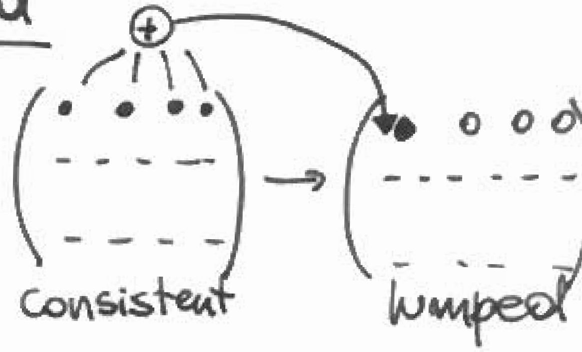
① $3M + 3m = 1$

②: different possibilities:

② Row (column) sum method

$$\boxed{M^e_{iaia} = \sum_{k=1}^{d} \sum_{b=1}^{n} M_{iakb}}$$

(no summation)



There is actually no need to compute the consistent mass matrix first since:

$$M_{iaia} = \sum_{k=1}^{d} \sum_{b=1}^{d} \int_{\Omega_0} \rho_0 \, \delta_{ik} \, N^e_a N^e_b \, dV$$

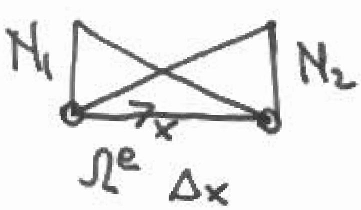
$$\sum_{k=1}^{d} \delta_{ik} = 1 \qquad \sum_{b=1}^{n} N^e_b = 1$$

$$M_{iaia} = \int_{\Omega^e_0} \rho_0 \, N^e_a \, dV \qquad \text{(independent of "i")}$$

Preserves the total mass.

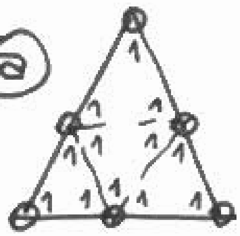
$$\boxed{\underset{\text{(no sum)}}{M_{iaia}} = \int_{\Omega^e_0} \rho_0 \, N^e_a \, dV = \sum_q \rho_0 \, N_{a(\xi_q)} \, w_q}$$

Example:



$$N_2 = \frac{x}{\Delta x} \, , \quad N_1 = 1 - \frac{x}{\Delta x}$$

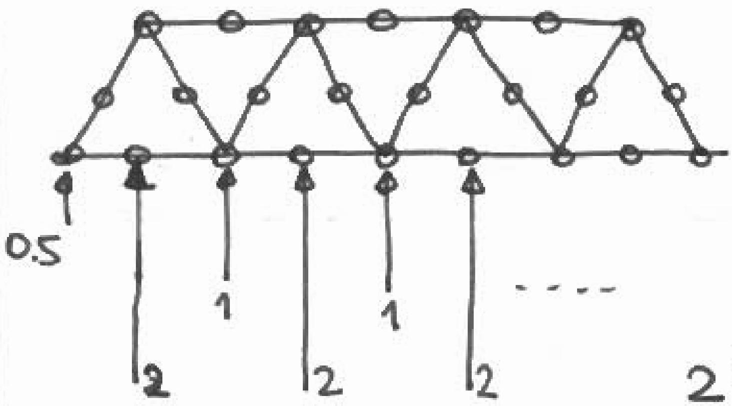
(a)



$M = 3m \Rightarrow m = \frac{1}{12} \quad M = \frac{3}{12} = \frac{1}{4}$

$$\boxed{M^{lumped} = \frac{\rho_0 A^e}{12}\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 3 & & \\ & & & & 3 & \\ & & & & & 3 \end{pmatrix}}$$

(b) "constant" accelerations to constant (uniform) contact forces



(1) $= 3M + 3m = 1$

(2) From:

$a_{midnode} = a_{corner}$

$$2\left\{\frac{f_{midnode}}{M} = \underbrace{\frac{f_{corner}}{3m}}_{\text{assumes mesh regular}}\right\}1$$

$\Rightarrow$ (2) $M = 6m$

$3 \times 6m + 3m = 1 \Rightarrow m = 1/21 \quad M = 6/21$

$$\boxed{M^{lumped} = \frac{\rho_0 A^e}{21}\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 6 & & \\ & & & & 6 & \\ & & & & & 6 \end{pmatrix}}$$

Others: Hughes Textbook, convergence properties

Algorithm analysis

Under what conditions does an algorithm yield convergent approximations?

General Initial Value problem (IVP):

$$\boxed{A\dot{y} + By = b(t) \quad , \; y(0) = y_0}$$

Examples:

- Heat conduction: $C\dot{\theta} + K\theta = f(t), \; \theta(0) = \theta_0$

identification: $y \equiv \theta \; , \; A \equiv C \; , \; B \equiv K, \; b \equiv f$

- Dynamics: $\begin{cases} M\ddot{x} + C\dot{x} + Kx = f(t) \\ x(0) = x_0 \; , \; \dot{x}(0) = \dot{x}_0 = v_0 \end{cases}$

Turn second order system into two first order coupled equations:

(1) $\dot{x} = v$ (2) $M\dot{v} + Cv + Kx = f$

$$y \equiv \begin{Bmatrix} x \\ v \end{Bmatrix}, \quad \begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix} \frac{d}{dt} \begin{Bmatrix} x \\ v \end{Bmatrix} + \begin{pmatrix} 0 & -I \\ K & C \end{pmatrix} \begin{Bmatrix} x \\ v \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix}$$

Multiply first equation by "K":

$$\underbrace{\begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix}}_{A} \underbrace{\frac{d}{dt} \begin{Bmatrix} x \\ v \end{Bmatrix}}_{\dot{y}} + \underbrace{\begin{pmatrix} 0 & -K \\ K & C \end{pmatrix}}_{B} \underbrace{\begin{Bmatrix} x \\ v \end{Bmatrix}}_{y} = \underbrace{\begin{Bmatrix} 0 \\ f \end{Bmatrix}}_{b}$$

Regularity assumptions: "A" symmetric positive definite

$$A = A^T, \quad A > 0$$

Energy norm: $\|y\|_E^2 = y^T A y$

- Example: Dynamics

$$\{x \quad v\} \begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix} \begin{Bmatrix} x \\ v \end{Bmatrix} =$$

$$\underbrace{x^T K x}_{\substack{2U \\ \text{2x strain energy}}} + \underbrace{v^T M v}_{\substack{2K \\ \text{2x kinetic energy}}}$$