16.21 Techniques of Structural Analysis and Design
Spring 2005
Section 2 - Energy and Variational Principles
Unit #7 - Concepts of work and energy

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Work

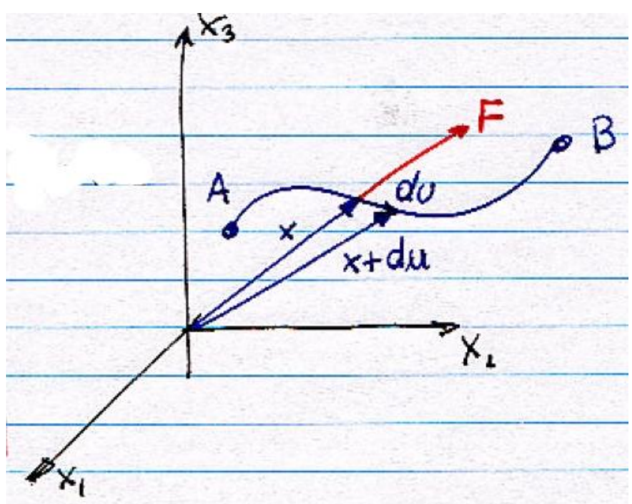


Figure 1: Work of a force on a moving particle

- Work done by a force:

$$dW = \mathbf{f} \cdot d\mathbf{u} = f_i u_i = \|\mathbf{f}\| \, \|d\mathbf{u}\| \cos(\widehat{\mathbf{f}\mathbf{u}}) \tag{1}$$

$$\boxed{W_{AB} = \int_A^B dW = \int_A^B \mathbf{f} \cdot d\mathbf{u}} \tag{2}$$

- Work done by a moment:

$$dW = \mathbf{M} \cdot d\theta = M_i \theta_i \tag{3}$$

$$\boxed{W_{AB} = \int_A^B dW = \int_A^B \mathbf{M} \cdot d\theta} \tag{4}$$

- Extend definition to material bodies: total work is the addition of the work done on all particles:
  - by forces distributed over the volume:

    $$W = \int_V \mathbf{f} \cdot \mathbf{u} dV$$

  - by forces distributed over the surface:

    $$W = \int_S \mathbf{t} \cdot \mathbf{u} dS$$

  - by concentrated forces:

    $$W = \sum_{i=1}^{n} \mathbf{f}_i \cdot \mathbf{u}(\mathbf{x}_i)$$

Another classification:

- Work done by external forces: we will assume that external forces don't change during the motion or deformation, i.e., they are independent of the displacements. This will lead to the potential character of the external work and to the definition of the *potential of the external forces* as the negative of the work done by the external forces.

- Work done by internal forces: the internal forces do depend on the deformation.

  In general, the work done by external forces and the work done by internal forces don't match (we saw that part of the work changes the kinetic energy of the material).

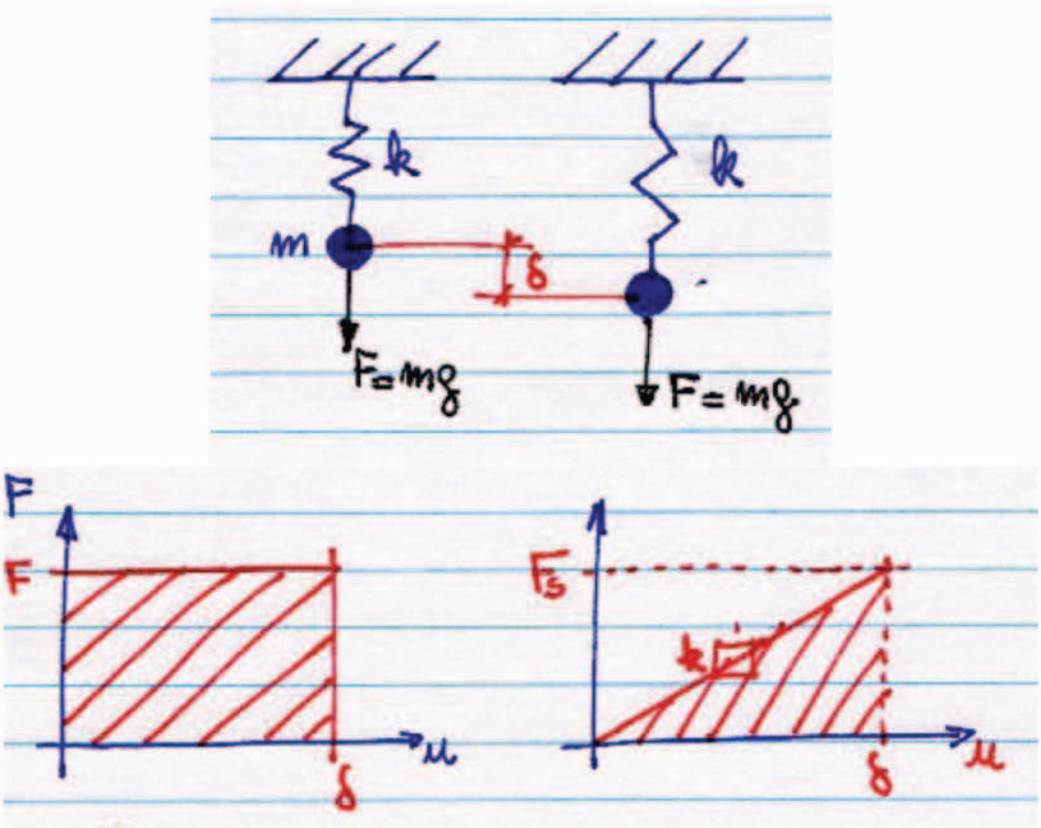


Figure 2: Spring loaded with a constant force

Example: Consider the following spring loaded with a constant force:

$$
\begin{align}
W_E &= F\delta,\ F \text{ doesn't change when } u \text{ goes from 0 to } \delta \tag{5}\\
&= mg\delta \tag{6}\\
W_I &= \int_0^\delta F_s(u)du,\ F_S : \text{ force on spring} \tag{7}\\
&= \int_0^\delta kudu = \frac{1}{2}k\delta^2 \tag{8}\\
&\Rightarrow \boxed{W_E \neq W_I} \tag{9}
\end{align}
$$

Remarks:

- $W_E = W_I$ would imply $\delta = 2\frac{mg}{k}$, which contradicts equilibrium: $\delta = \frac{mg}{k}$,
- before the final displacement δ is reached the system is not in equilibrium. How can you explain this?

Strain energy and strain energy density

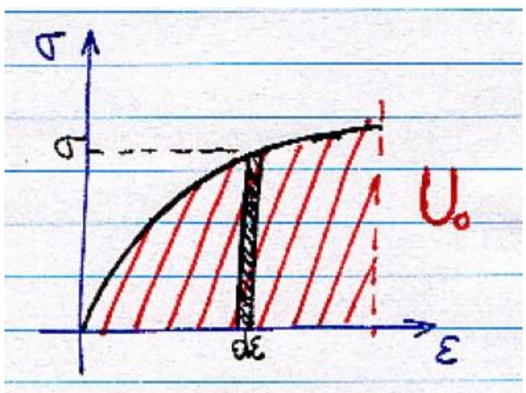


Figure 3: Strain energy density

Strain energy and strain energy density (see also unit on first law of thermodynamics):

$$U = \int_V U_0 dV \tag{10}$$

From first law:

$$\frac{\partial U_0}{\partial t} = \sigma_{ij}\dot{\epsilon}_{ij}$$

$$\underbrace{\frac{\partial U_0}{\partial \epsilon_{ij}}\dot{\epsilon}_{ij}}_{????} = \sigma_{ij}\dot{\epsilon}_{ij}$$

$$\boxed{\sigma_{ij} = \frac{\partial U_0}{\partial \epsilon_{ij}}}$$

$$\boxed{U_0 = \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij}}, \text{ not necessarily linear elastic}$$

Linear case:

$$U_0 = \int_0^{\epsilon_{ij}} C_{ijkl}\epsilon_{kl} d\epsilon_{ij} = \frac{1}{2}C_{ijkl}\epsilon_{kl}\epsilon_{ij} = \frac{1}{2}\sigma_{ij}\epsilon_{ij} \tag{11}$$

Complementary strain energy and complementary strain energy density

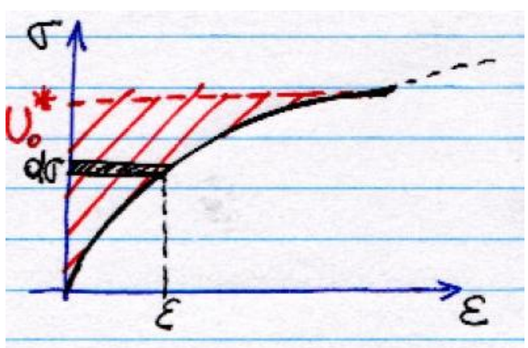


Figure 4: Complementary strain energy density

$$U^\star = \int_V U_0^\star dV \tag{12}$$

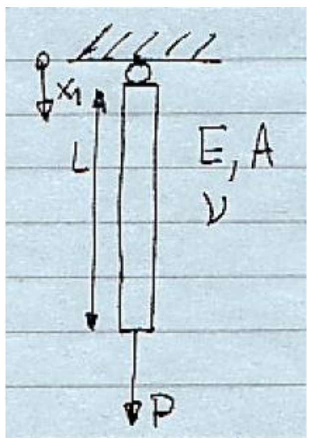
$$\boxed{U_0^\star = \int_0^{\sigma_{ij}} \epsilon_{ij} d\sigma_{ij}} \tag{13}$$

Linear case: $\epsilon_{ij} = S_{ijkl}\sigma_{kl}$, where $S_{ijkl} = C^{-1}_{ijkl}$

$$U_0^\star = \int_0^{\sigma_{ij}} S_{ijkl}\sigma_{kl} d\sigma_{ij} = \frac{1}{2} S_{ijkl}\sigma_{kl}\sigma_{ij} = \frac{1}{2}\epsilon_{ij}\sigma_{ij} \tag{14}$$

$$\Rightarrow \boxed{U_0^\star = U_0} \text{ for a linear elastic material} \tag{15}$$

Example: Compute the strain energy density, strain energy, and their complementary counterparts for the linear elastic bar loaded axially shown in the figure:



$$U_0 = \int_0^{\epsilon_0} \sigma_{11} d\epsilon_{11} + \int_0^{-\nu\epsilon_0} \sigma_{22} d\epsilon_{22} + \dots$$
$$= \int_0^{\epsilon_0} E\epsilon_{11} d\epsilon_{11} = \frac{1}{2} E\epsilon_0^2$$

From equilibrium we know: $\sigma_0 = \frac{P}{A}$.
From the constitutive law: $\epsilon_0 = \frac{\sigma_0}{E} = \frac{P}{AE}$

$$\Rightarrow U_0 = \frac{1}{2}\frac{P^2}{EA^2}$$
$$U = \int_V U_0 dV = \frac{ALP^2}{2EA^2} = \frac{P^2 L}{2EA}$$

$$
\begin{aligned}
U_0^\star &= \int_0^{\sigma_0} \epsilon_{11} d\sigma_{11} + \int_0^0 \epsilon_{22} d\sigma_{22} + \ldots \\
&= \int_0^{\sigma_0} \frac{\sigma_{11}}{E} d\sigma_{11} = \frac{1}{2E}\sigma_{11}^2 \Big|_0^{\sigma_0} = \frac{\sigma_0^2}{2E} = \frac{P^2}{2EA^2} = U_0!! \\
U^\star &= \int_V U_0^\star dV = \frac{ALP^2}{2EA^2} = \frac{P^2 L}{2EA} = U!!
\end{aligned}
$$

Potential Energy

Capacity of the system (material body + external forces) to return work

$$\boxed{\Pi = U + V}, V: \text{ potential of external loads} \tag{16}$$

$$\boxed{V = -\int_S \bar{t}_i u_i dS - \int_V \bar{f}_i u_i dV} \tag{17}$$

$$\boxed{\Pi = \int_V \frac{1}{2}\sigma_{ij}\epsilon_{ij} dV - \int_S \bar{t}_i u_i dS - \int_V \bar{f}_i u_i dV} \tag{18}$$

This expression applies to linear elastic materials (why?).