# Unit 22
# Vibration of Multi Degree-Of-Freedom Systems

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Previously saw (in Unit 19) that a multi degree-of-freedom system has the same basic form of the governing equation as a single degree-of-freedom system.

The difference is that it is a matrix equation:

$$\underset{\sim}{m}\,\underset{\sim}{\ddot{q}} + \underset{\sim}{k}\,\underset{\sim}{q} = \underset{\sim}{F} \qquad (22\text{-}1)$$

$\sim$ = matrix

So apply the same solution technique as for a single degree-of-freedom system. Thus, first deal with…

## Free Vibration

Do this by again setting forces to zero:

$$\underset{\sim}{F} = \underset{\sim}{0}$$

$$\underset{\sim}{m}\,\underset{\sim}{\ddot{q}} + \underset{\sim}{k}\,\underset{\sim}{q} = \underset{\sim}{0} \qquad (22\text{-}2)$$

Again assume a solution which has harmonic motion. It now has multiple components:

$$\underset{\sim}{q}(t) = \underset{\sim}{A} e^{i\omega t} \qquad (22\text{-}3)$$

where ω are the natural frequencies of the system
and:

$\underset{\sim}{A}$ is a vector of constants $= \begin{Bmatrix} \vdots \\ A_i \\ \vdots \end{Bmatrix}$

Substituting the assumed solution into the matrix set of governing equations:

$$\Rightarrow -\omega^2 \underset{\sim}{m} \underset{\sim}{A} e^{i\omega t} + \underset{\sim}{k} \underset{\sim}{A} e^{i\omega t} = \underset{\sim}{0}$$

To be true for all cases:

$$\left[\underset{\sim}{k} - \omega^2 \underset{\sim}{m}\right] \underset{\sim}{A} = \underset{\sim}{0} \qquad (22\text{-}4)$$

This is a standard ***eigenvalue*** problem.

Either:

$\underset{\sim}{A} = 0$ (trivial solution)

or

The determinant:

$$\left| \underset{\sim}{k} - \omega^2 \underset{\sim}{m} \right| = 0 \qquad (22\text{-}5)$$

There will be n eigenvalues for an n degree-of-freedom system. In this case:

eigenvalue = natural frequency

⇒ **n** degree-of-freedom system has **n** natural frequencies

Corresponding to each eigenvalue (natural frequency), there is an…

Eigenvector -- Natural Mode

- Place natural frequency $\omega_r$ into equation (22-4):

$$\left[ \underset{\sim}{k} - \omega_r^2 \underset{\sim}{m} \right] \underset{\sim}{A} = \underset{\sim}{0}$$

- Since determinant = 0, there is one dependent equation, so one cannot solve explicitly for $\underset{\sim}{A}$. However, one can solve for the relative values of the components of $\underset{\sim}{A}$ in terms of (normalized by) one component

- Say divide through by $A_n$:

$$\left[\underset{\sim}{k} - \omega_i^2\, \underset{\sim}{m}\right] \begin{Bmatrix} \vdots \\ A_i / A_n \\ \vdots \\ 1 \end{Bmatrix} = \underset{\sim}{0}$$

- Solve for $A_i$ / $A_n$ for each $\omega_r$

- Call the eigenvector $\begin{Bmatrix} \vdots \\ A_i / A_n \\ \vdots \\ 1 \end{Bmatrix} = \underset{\sim}{\phi}_i^{(r)} \leftarrow$ Indicates solution for $\omega_r$

- Do this for each eigenvalue

frequency: $\omega_1$, $\omega_2$ ……. $\omega_n$

associated mode: $\underset{\sim}{\phi}_i^{(1)}$ $\underset{\sim}{\phi}_i^{(2)}$ $\underset{\sim}{\phi}_i^{(n)}$

For each eigenvalue, the homogeneous solution is:

$$\underset{\sim}{q}_{i\,\text{hom}} = \underset{\sim}{\phi}_i^{(r)} e^{i\omega_r t} = C_1 \underset{\sim}{\phi}_i^{(r)} \sin\omega_r t + C_2 \underset{\sim}{\phi}_i^{(r)} \cos\omega_r t$$

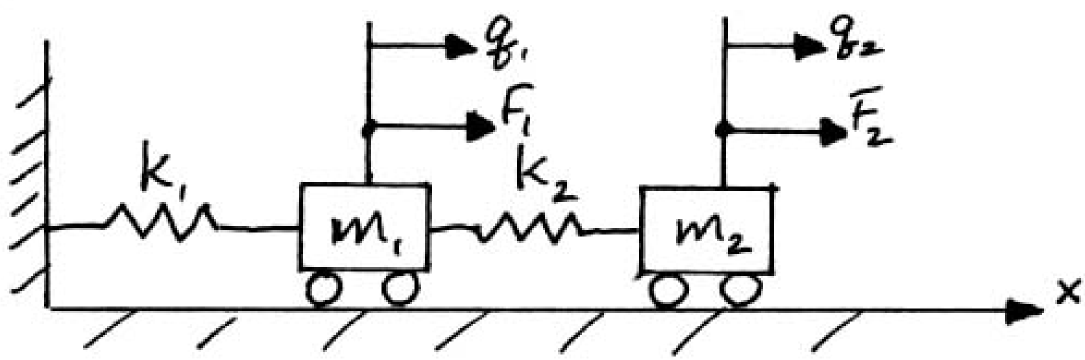
*homogeneous*

Still an undetermined constant in each case ($A_n$) which can be determined from the Initial Conditions

- Each homogeneous solution physically represents a possible free vibration mode
- Arrange natural frequencies from lowest ($\omega_1$) to highest ($\omega_n$)
- By superposition, any combinations of these is a valid solution

Example: Two mass system (from Unit 19)

***Figure 22.1*** **Representation of dual spring-mass system**



The governing equation was:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Thus, from equation (22-5):

$$\begin{vmatrix} (k_1 + k_2) - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{vmatrix} = 0$$

This gives:

$$\left[(k_1 + k_2) - \omega^2 m_1\right]\left[k_2 - \omega^2 m_2\right] - k_2^{\,2} = 0$$

This leads to a quadratic equation in $\omega^2$. Solving gives two roots ($\omega_1^2$ and $\omega_2^2$) and the natural frequencies are $\omega_1$ and $\omega_2$

Find the associated eigenvectors in terms of $A_2$ (i.e. normalized by $A_2$)

Go back to equation (22-4) and divide through by $A_2$:

$$\begin{bmatrix} (k_1 + k_2) - \omega_r^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega_r^2 m_2 \end{bmatrix} \begin{Bmatrix} A_1 \\ 1 \end{Bmatrix} = 0$$

*Normalized constant*

$$\Rightarrow A_1 = \frac{k_2}{k_1 + k_2 - \omega_r^2 m_1} \quad \text{for } \omega_r \text{ mode}$$

Thus the eigenvectors are:

$$\underset{\sim}{\phi_i}^{(1)} = \begin{Bmatrix} \dfrac{k_2}{k_1 + k_2 - \omega_1^2 m_1} \\ 1 \end{Bmatrix} \qquad \underset{\sim}{\phi_i}^{(2)} = \begin{Bmatrix} \dfrac{k_2}{k_1 + k_2 - \omega_2^2 m_1} \\ 1 \end{Bmatrix}$$

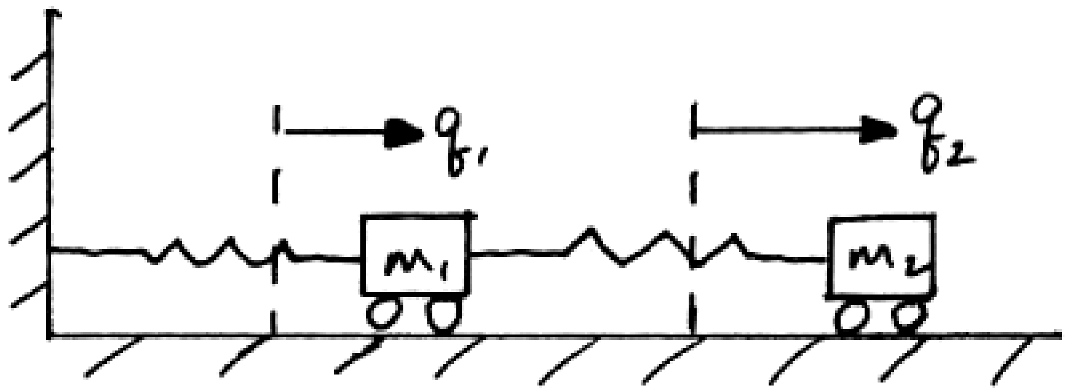
For the case of Initial Conditions of 0, the cos term goes away and are left with…

$$\underset{\sim}{q}(t) = \underset{\sim}{\phi}^{(r)} \sin \omega_r t$$

<u>Physically</u> the modes are:

## *Figure 22.2* Representation of modes of spring-mass system

$\phi^{(1)}$




$\omega_1$ **(lowest frequency)**

**masses move in same direction**

$\phi^{(2)}$


$\omega_2$ **(higher frequency)**

**masses move in opposite direction**

General Rules for discrete systems:

- Can find various modes (without amplitudes) by considering combinations of positive and negative (relative) motion.

  However, be careful of (-1) factor across entire mode.

  For example, in two degree-of-freedom case

$$\left.\begin{array}{c} +\ + \\ \\ -\ - \end{array}\right\} \text{ same mode} \qquad \left.\begin{array}{c} +\ - \\ \\ -\ + \end{array}\right\} \text{ same mode}$$

- The more “reversals” in direction, the higher the mode (and the frequency)
- It is harder to excite higher modes

This can be better illustrated by considering the vibration of a beam. So look at:

## Representation of a Beam as a Discrete Mass System

How?

- Lump mass into discrete locations with constraint that total mass be the same
- Connect masses by rigid connections with rotational springs at each mass
- Stiffnesses of connections are influence coefficients (dependent on locations of point masses)
- Forces applied to point masses

So:

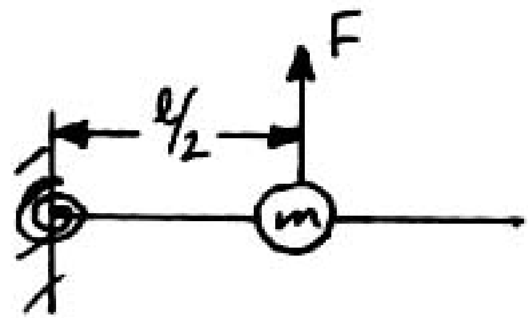
*Figure 22.3* **Representation of cantilevered beam as single mass system**


where: $m = \rho t w \ell$ ← length

density, thickness, width

Could also put mass at mid-point:

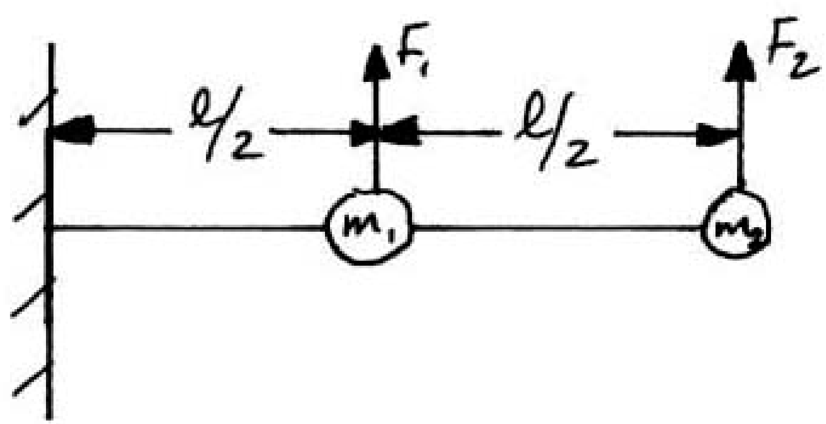
***Figure 22.4*** **Representation of cantilevered beam as mid-point mass system**




⇒ get a different representation

Consider the next complicated representation (simplest multi-mass/degree-of-freedom system)

***Figure 22.5*** **Representation of cantilevered beam as dual spring-mass system**



each m is one half of total mass of beam for constant cross-section case

Use influence coefficients, get $\underset{\sim}{C}$ matrix, invert to get $\underset{\sim}{K}$. Resulting equation is:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Same form as before, so solution takes same form. For initial rest conditions:

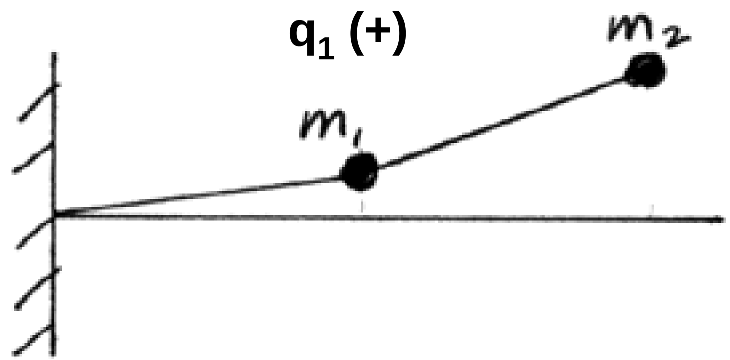
$$\underset{\sim}{q}(t) = \underset{\sim}{\phi}^{(r)} \sin \omega_r t$$

Have two eigenvalues (natural frequencies) and associated eigenvectors (modes)

⇒ Modes have clear physical interpretation here:

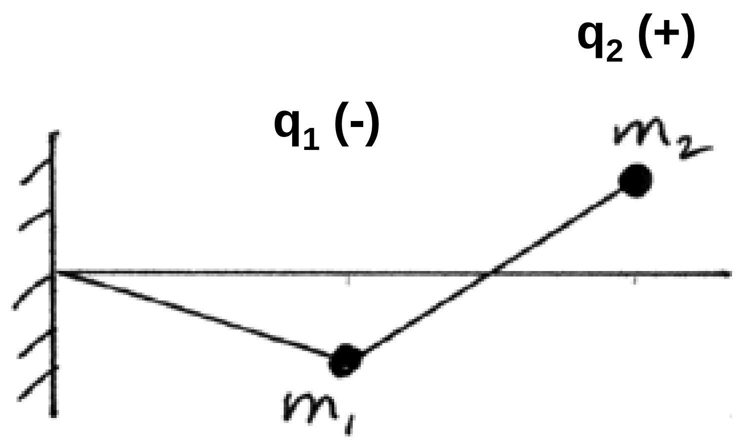
***Figure 22.6*** **Representation of deflection modes of cantilevered beam as dual spring-mass system**

$\phi^{(1)}$




**$\omega_1$ (lowest frequency)**

$\phi^{(2)}$




**$\omega_2$ (higher frequency)**

Can extend by dividing beam into more discrete masses

--> get better representation with more equations but same basic treatment/approach

In considering the modes that result from such an analysis, there is a key finding:

**Orthogonality Relations**

It can be shown that the modes of a system are orthogonal. That is:

*transpose*

$$\underset{\sim}{\phi}^{(r)^T} \underset{\sim}{m}\, \underset{\sim}{\phi}^{(s)} = 0 \qquad (22\text{-}6)$$

for r ≠ s

If r = s, then a finite value results:

$$\underset{\sim}{\phi}^{(r)^T} \underset{\sim}{m}\, \underset{\sim}{\phi}^{(r)} = M_r \qquad (22\text{-}7)$$

some value

So the general relation for equations (22-6) and (22-7) can be written as:

$$\boxed{\underset{\sim}{\phi}^{(r)^T} \underset{\sim}{m} \, \underset{\sim}{\phi}^{(s)} = \delta_{rs} M_r} \qquad (22\text{-}8)$$

$\delta_{rs}$ is the kronecker delta where:

$\delta_{rs} = 0$ for $r \neq s$
$\delta_{rs} = 1$ for $r = s$

This relation allows the transformation of the governing equation into a special set of equations based on the (normal) nodes…

## Normal Equations of Motion

These resulting equations are uncoupled and thus much easier to solve

The starting point is the eigenvectors (modes) and the orthogonality relations

One must also note that:

$$\underset{\sim}{\phi}^{(r)^T} \underset{\sim}{k} \, \underset{\sim}{\phi}^{(s)} = \delta_{rs} M_r \, \omega_r^{\,2} \qquad (22\text{-}9)$$

(can show using equations (22-2) and (22-8) )

Have shown that the homogeneous solution to the general equation:

$$\underset{\sim}{m}\,\ddot{\underset{\sim}{q}} + \underset{\sim}{k}\,\underset{\sim}{q} = \underset{\sim}{F} \qquad (22\text{-}1)$$

is the sum of the eigenvectors (modes):

$$\underset{\sim}{q_i}(t) = \sum_{r=1}^{n} \phi_i^{(r)}\,\xi_r(t) \qquad (22\text{-}10)$$

n = number of degrees of freedom

Where $\xi_r(t)$ is basically a magnitude associated with the mode $\phi^{(r)}$ at time t. The $\xi_r$ become the *"normalized coordinates".*

Thus:

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_n \end{Bmatrix} = \begin{bmatrix} \phi_1^{(1)} & \phi_1^{(2)} & \phi_1^{(3)} & \cdots & \phi_1^{(n)} \\ \phi_2^{(2)} & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & \cdots & \phi_n^{(n)} \end{bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \vdots \\ \xi_n \end{Bmatrix}$$

which can be written as:

$$\underset{\sim}{q} = \underset{\sim}{\phi}\,\underset{\sim}{\xi} \qquad (22\text{-}11)$$

Placing (22-11) into (22-1)

$$\underset{\sim}{m}\underset{\sim}{\phi}\ddot{\underset{\sim}{\xi}} + \underset{\sim}{k}\underset{\sim}{\phi}\underset{\sim}{\xi} = \underset{\sim}{F} \qquad (22\text{-}12)$$

Now multiply this equation by the transpose of $\underset{\sim}{\phi}$:

$$\underset{\sim}{\phi}^T = \begin{bmatrix} \phi_1^{(1)} & \phi^{(2)} & \cdots & \cdots \\ \phi_2^{(2)} & \cdots & & \\ \vdots & & & \\ \phi^{(n)} & \cdots & \cdots & \cdots \end{bmatrix}$$

$$\underset{\sim}{\phi}^T\underset{\sim}{m}\underset{\sim}{\phi}\ddot{\underset{\sim}{\xi}} + \underset{\sim}{\phi}^T\underset{\sim}{k}\underset{\sim}{\phi}\underset{\sim}{\xi} = \underset{\sim}{\phi}^T\underset{\sim}{F} \qquad (22\text{-}13)$$

Notice that the terms of $\underset{\sim}{\phi}^T\underset{\sim}{m}\,\underset{\sim}{\phi}$ and $\underset{\sim}{\phi}^T\underset{\sim}{k}\underset{\sim}{\phi}$ will result in most of the terms being zero due to the orthogonality relation (equation 22-8). Only the diagonal terms will remain.

Thus, (22-13) becomes a set of <u>uncoupled</u> equations: (via 22-8 and 22-9)

$$\boxed{M_r \ddot{\xi}_r + M_r \omega_r^{\,2} \xi_r = \Xi_r} \qquad (22\text{-}14)$$

r = 1, 2 …n

That is:

$$M_1 \ddot{\xi}_1 + M_1 \omega_1^{\,2} \xi_1 = \Xi_1$$

$$M_2 \ddot{\xi}_2 + M_2 \omega_2^{\,2} \xi_2 = \Xi_2$$

$$M_n \ddot{\xi}_n + M_n \omega_n^{\,2} \xi_n = \Xi_n$$

where:

$$M_r = \begin{bmatrix} \phi_1^{(r)} & \phi_2^{(r)} & \cdots \end{bmatrix} \underset{\sim}{m} \begin{Bmatrix} \phi_1^{(r)} \\ \phi_2^{(r)} \\ \vdots \end{Bmatrix}$$ = Generalized mass of rth mode

and:

$$\Xi_r = \begin{bmatrix} \phi_1^{(r)} & \phi_2^{(r)} & \cdots \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \end{Bmatrix} = \text{Generalized force of rth mode}$$

$\xi_r(t)$ = normal coordinates

The equations have been transformed to normal coordinates and are now uncoupled single degree-of-freedom systems

<u>Implication</u>: Each equation can be solved **<u>separately</u>**

The overall solution is then a superposition of the individual solutions (normal nodes)

## <u>Free Vibration</u> $(\Xi = 0)$

--> <u>solution</u>…use same technique as before

- For any equation r:

$$\xi_r = a_r \sin \omega_r t + b_r \cos \omega_r t$$

- Get $a_r$ and $b_r$ via *transformed* Initial Conditions

$$\xi_r(0) \;=\; \frac{1}{M_r}\underset{\sim}{\phi}^{(r)T}\underset{\sim}{m}\; q_i(0) \;=\; b_r$$

$$\dot{\xi}_r(0) \;=\; \frac{1}{M_r}\underset{\sim}{\phi}^{(r)T}\underset{\sim}{m}\; \dot{q}_i(0) \;=\; a_r\omega_r$$

Notes:

- KEY SIMPLIFICATION is that often only first few (lowest) modes are excited so can solve only first few equations. Can add more modes (equations) to improve solution if needed.

- This is a rigorous treatment -- no approximation made by going to normal coordinates.

But, this has all been based on the homogeneous case (free vibration), what about…

# Forced Vibration $(\Xi \neq 0)$

Response is still made up of the natural modes. Solution is found using the same approach as for a single degree-of-freedom system…

- Break up each generalized force, $\Xi_r$, into a series of impulses
- Use Duhamel's (convolution) integral to get response for each degree of freedom
- Stay in normalized coordinates

The solution for any mode will thus look like:

$$\xi_r(t) = \frac{1}{M_r \omega_r} \int_0^t \Xi_r(\tau) \sin \omega_r (t - \tau) \, d\tau$$

and equation (22-11) then gives:

$$q_i(t) = \sum_{r=1}^{n} \phi_i^{(r)} \xi_r(t)$$

Again, use Initial Conditions to get constants

Exact same procedure as single degree-of-freedom system. Do it multiple times and add up.

**(Linear ⇒ Superposition)**

Can therefore represent any system by discrete masses. As more and more discrete points are taken, get a better model of the actual behavior. Taking this to the limit will allow the full representation of the behavior of continuous systems.