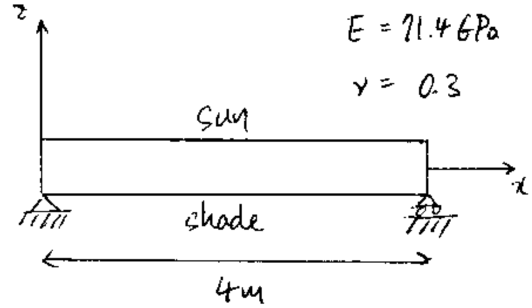
Practice Problems

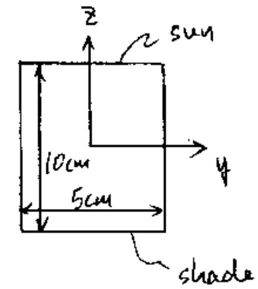
5.



$E = 71.4\,GPa$ $\quad \alpha = 7.2\ \mu\epsilon/^\circ C$

$\nu = 0.3$ $\quad T_{zero} = 0^\circ C$

zero stress temp.



Let's find the temperature distribution first. Assuming a quadratic distribution of the form,

$$T(z) = a + b(z + 0.05m)^2$$

and plugging in the given temperatures at the sun side and the shade side, we get:

$$T(-0.05m) = a = -90^\circ C$$

$$T(+0.05m) = -90^\circ C + b(0.1m)^2 = 80^\circ C$$

$$\Rightarrow b = 17000\ ^\circ C/m^2$$

$$\therefore \quad T(z) = -90^\circ C + 17000\ ^\circ C/m^2 (z + 0.05m)^2$$

$$\Delta T = T(z) - T_{zero} = T(z) - 0^\circ C$$

$$\Rightarrow \quad \Delta T = -90^\circ C + 17000\ ^\circ C/m^2 (z + 0.05m)^2 \quad \text{—— ①}$$

$$\text{or} \quad \Delta T = 17000\ ^\circ C/m^2\, z^2 + 1700\ ^\circ C/m\, z - 47.5^\circ C$$

ii) Determine the stress distribution in the beam.

Since y and z are the principal axes, the modulus-weighted equation with $I_{yz}^{*} = 0$ can be used. From the lecture notes, unit #14, p

$$\sigma_{xx} = \frac{E}{E_1}\left[\frac{F^{TOT}}{A^{*}} - \frac{M_z^{TOT}}{I_z^{*}}y - \frac{M_y^{TOT}}{I_y^{*}}z - E_1\alpha\Delta T\right] \quad \text{— ②}$$

The beam is made of one material, so the modulus-weighted properties are equal to the "regular" properties.

$$E_1 = E = 71.4\,GPa \Rightarrow \frac{E}{E_1} = 1$$

$$A^{*} = A, \quad I_z^{*} = I_z, \quad I_y^{*} = I_y.$$

Thus, equation ② is

$$\sigma_{xx} = \frac{F^{TOT}}{A} - \frac{M_z^{TOT}}{I_z}y - \frac{M_y^{TOT}}{I_y}z - E\alpha\Delta T \quad \text{— ③}$$

We also know that

$$F^{TOT} = F^{M} + \iint E\alpha\Delta T\,dA \quad \text{— ④}$$

$$M_z^{TOT} = M_z^{M} - \iint E\alpha\Delta T\,y\,dA \quad \text{— ⑤}$$

$$M_y^{TOT} = M_y^{M} - \iint E\alpha\Delta T\,z\,dA \quad \text{— ⑥}$$

Since there are no mechanical loads applied to the beam,

$$F^{M} = M_z^{M} = M_y^{M} = 0 \quad \text{———} \quad ①$$

Then,

$$F^{TOT} = \iint E\alpha \Delta T \, dA$$

$$\Rightarrow F^{TOT} = E\alpha \int_{-0.025m}^{0.025m} \int_{-0.05m}^{0.05m} \left[-90^\circ C + 17{,}000^\circ C/m^2 (z + 0.05m)^2\right] dz\, dy$$

$$= E\alpha \int_{-0.025}^{0.025m} \left[-90z + \frac{1}{3} 17000 (z + 0.05m)^3\right]_{-0.05m}^{0.05m} dy$$

$$= E\alpha \,[-3.33](0.5)$$

$$\therefore \quad F^{TOT} = -0.167\, E\alpha \quad (N)$$

$$\Rightarrow \quad F^{TOT} = -85700 \, N$$

$$M_z^{TOT} = -\iint E\alpha \Delta T y \, dA$$

$$= -E\alpha \int_{-0.05m}^{0.05m} \int_{-0.025m}^{0.025m} \left[-90^\circ C + 17000^\circ C/m^2 (z + 0.05m)^2\right] y \, dy\, dz$$

$$= 0$$

$$\therefore \quad M_z^{TOT} = 0$$

$$M_y^{TOT} = -\iint E\alpha \Delta T z \, dA$$

$$= -E\alpha \int_{-0.05m}^{0.05m} \int_{-0.025m}^{0.025m} \left[-90^\circ C + 17000^\circ C/m^2 (z + 0.05m)^2\right] z \, dy\, dz$$

$$= -E\alpha \int_{-0.025m}^{0.025m} \left[-\frac{1}{2} 90 z^2 + \frac{1}{4} 17000 z^4 + \frac{1}{3} 1700 z^3 + \frac{42.5}{2} z^2\right]_{-0.05m}^{0.05m} dy$$

$$= -E\alpha \int_{-0.025m}^{0.025m} 0.142\, dy$$

$$\therefore \quad M_y^{TOT} = -0.0071\, E\alpha$$

$$\Rightarrow \quad M_y^{TOT} = -3640 \ \text{Nm}$$

Rewriting equation ⑤ using these values, we get,

$$\sigma_{xx} = \underbrace{-\frac{0.167\, E\alpha}{A} - E\alpha\Delta T}_{\text{axial contribution to stress}} + \underbrace{\frac{0.0071\, E\alpha z}{I_y}}_{\text{bending contribution to stress}} \qquad \text{——— ⑥}$$

Since

$$A = (0.1m)(0.05m) = 0.005m^2$$

$$I_y = \frac{1}{12}(0.05m)(0.1m)^3 = 4.17 \times 10^{-6} m^4$$

equation ⑥ is

$$\sigma_{xx} = E\alpha\left[-\frac{0.167}{0.005} - \left(-90^\circ + 17000^\circ/m^2 (z + 0.05m)^2\right) + \frac{0.0071 z}{4.17 \times 10^{-6}}\right]$$

$$\Rightarrow \sigma_{xx} = 5.14 \times 10^5 \left[56.6 - 17000(z + 0.05m)^2 + 1700 z\right]$$

$$\therefore \boxed{\sigma_{xx} = -8740 z^2 + 7.25 \quad \text{MPa}} \ (z \text{ in meters})$$

* Note : another method to solve the problem is to use the following equation

① $\Sigma F = 0 \Rightarrow \int_{-0.05m}^{0.05m} \sigma_{xx}\, dz = 0$

② $\Sigma M = 0 \Rightarrow \int_{-0.05m}^{0.05m} \sigma_{xx} z\, dz = 0$

③ $\epsilon_{xx}^{TOT} = \alpha\Delta T + \frac{\sigma_{xx}}{E} \Rightarrow \sigma_{xx} = -E\alpha\Delta T + E\epsilon_{xx}^{TOT}$

④ $\epsilon_{xx}^{TOT} = \frac{du_o}{dx} - z\frac{d^2 w}{dx^2}$

solve for $\frac{du_o}{dx}$ & $\frac{d^2 w}{dx^2}$ which are functions of only x.

b) Find axial and vertical displacements of the beam as functions of x.

- $u_o(x)$ = axial displacement:

From unit #14, p ,

$$\frac{du_o}{dx} = \frac{F^{TOT}}{E_1 A^*}$$

$$\Rightarrow \frac{du_o}{dx} = \frac{F^{TOT}}{EA} = \frac{-0.167 E\alpha}{E(0.005m^2)} = -33.4\alpha$$

Integrating, we get

$$u_o(x) = -33.4\alpha x + C_1$$

The boundary condition at $x=0$ is

$$u_o(0) = 0 \Rightarrow C_1 = 0$$

$$\therefore \boxed{u_o(x) = (-2.40 \times 10^{-4})\, x \quad m} \quad (x \text{ in } m)$$

- $w(x)$ = vertical displacement :

From unit #14, p ,

$$-\frac{d^2 w}{dx^2} = \frac{-I_z^* M_y^{TOT} + I_{yz}^* M_z^{TOT}}{E_1 (I_y^* I_z^* - I_{yz}^{*2})}$$

$$\Rightarrow \frac{d^2 w}{dx^2} = \frac{M_y^{TOT}}{E I_y} \qquad (I_{yz}^* = 0, \; E_1 = E)$$

$$= \frac{-0.0071 E\alpha}{E(4.17 \times 10^{-6} m^4)}$$

$$= -1.23 \times 10^{-2}$$

Integrating twice, we get

$$w(x) = -6.13\times10^{-3}x^2 + C_1 x + C_2 \quad m$$

The boundary conditions at $x=0$ is

$$w(0) = 0 \quad \Rightarrow \quad C_2 = 0$$

$$w(4m) = 0 \quad \Rightarrow \quad 4C_1 = 9.81 \times 10^{-2}$$

$$C_1 = 0.024$$

$$\therefore \boxed{w(x) = -6.13\times10^{-3}x^2 + 0.024\,x \quad m} \quad (x \text{ in } m)$$

From the displacements, we can see that the bar decreases in length and bends upward due to the thermal condition.