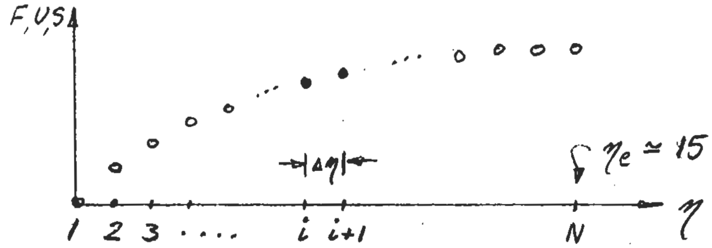
Solution of Falkner-Skan Equation by Finite Differences

$$\frac{\partial F}{\partial \eta} - U = 0$$

$$\frac{\partial U}{\partial \eta} - S = 0$$

$$\frac{\partial S}{\partial \eta} + \frac{1+\beta_u}{2} FS + \beta_u (1-U^2) = 0$$



$\eta = 0: \quad F = U = 0 \qquad \eta = \eta_e: \quad U = 1$

Discrete System: (Trapezoidal Formula)

$$F_{i+1} - F_i - \frac{\Delta\eta}{2}\left(U_{i+1} + U_i\right) \equiv R_{F_i} = 0$$

$$U_{i+1} - U_i - \frac{\Delta\eta}{2}\left(S_{i+1} + S_i\right) \equiv R_{U_i} = 0$$

$$S_{i+1} - S_i + \left(\frac{1+\beta_u}{2}\right)\frac{\Delta\eta}{2}\left(F_{i+1} S_{i+1} + F_i S_i\right) + \beta_u \Delta\eta\left(1 - \tfrac{1}{2}\left(U_{i+1}^2 + U_i^2\right)\right) \equiv R_{S_i} = 0$$

BC's: $\quad F_1 \equiv R_{BC_1} = 0 \qquad U_1 \equiv R_{BC_2} = 0 \qquad U_N - 1 \equiv R_{BC_3} = 0$

Newton-Raphson System for $\delta F_i \quad \delta U_i \quad \delta S_i$

$$\begin{array}{c}
\begin{array}{cccccccccc} \delta F_1 & \delta U_1 & \delta S_1 & \delta F_2 & \delta U_2 & \delta S_2 & \cdots & \delta F_N & \delta U_N & \delta S_N \end{array} \\
\begin{bmatrix}
\frac{\partial R_{BC_1}}{\partial F_1} & 0 & 0 & & 0 & & & & & & & & \\
0 & \frac{\partial R_{BC_2}}{\partial U_1} & 0 & & & & & & & & & & \\
\frac{\partial R_{S_1}}{\partial F_1} & \frac{\partial R_{S_1}}{\partial U_1} & \frac{\partial R_{S_1}}{\partial S_1} & \frac{\partial R_{S_1}}{\partial F_2} & \frac{\partial R_{S_1}}{\partial U_2} & \frac{\partial R_{S_1}}{\partial S_2} & & & & & & & \\
\frac{\partial R_{F_1}}{\partial F_1} & \frac{\partial R_{F_1}}{\partial U_1} & 0 & \frac{\partial R_{F_1}}{\partial F_2} & \frac{\partial R_{F_1}}{\partial U_2} & 0 & & & & & & & \\
0 & \frac{\partial R_{U_1}}{\partial U_1} & \frac{\partial R_{U_1}}{\partial S_1} & 0 & \frac{\partial R_{U_1}}{\partial U_2} & \frac{\partial R_{U_1}}{\partial S_2} & & & & & & & \\
 & 0 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\
 & & & & & & & & & & 0 & & \\
 & & & & & & & \frac{\partial R_{S_{N-1}}}{\partial F_{N-1}} & \frac{\partial R_{S_{N-1}}}{\partial U_{N-1}} & \frac{\partial R_{S_{N-1}}}{\partial S_{N-1}} & \frac{\partial R_{S_{N-1}}}{\partial F_N} & \frac{\partial R_{S_{N-1}}}{\partial U_N} & \frac{\partial R_{S_{N-1}}}{\partial S_N} \\
 & & & & & & & \frac{\partial R_{F_{N-1}}}{\partial F_{N-1}} & \frac{\partial R_{F_{N-1}}}{\partial U_{N-1}} & 0 & \frac{\partial R_{F_{N-1}}}{\partial F_N} & \frac{\partial R_{F_{N-1}}}{\partial U_N} & 0 \\
 & & & & & & & 0 & \frac{\partial R_{U_{N-1}}}{\partial U_{N-1}} & \frac{\partial R_{U_{N-1}}}{\partial S_{N-1}} & 0 & \frac{\partial R_{U_{N-1}}}{\partial U_N} & \frac{\partial R_{U_{N-1}}}{\partial S_N} \\
 & & & & & & & & 0 & & 0 & \frac{\partial R_{BC_3}}{\partial U_N} & 0
\end{bmatrix}
\end{array}
\begin{bmatrix} \delta F_1 \\ \delta U_1 \\ \delta S_1 \\ \delta F_2 \\ \delta U_2 \\ \delta S_2 \\ \vdots \\ \delta F_N \\ \delta U_N \\ \delta S_N \end{bmatrix}
+ \delta\beta_u
\begin{bmatrix} 0 \\ 0 \\ \frac{\partial R_{S_1}}{\partial \beta_u} \\ 0 \\ 0 \\ \frac{\partial R_{S_2}}{\partial \beta_u} \\ \vdots \\ \frac{\partial R_{S_{N-1}}}{\partial \beta_u} \\ 0 \\ 0 \\ 0 \end{bmatrix}
= -
\begin{bmatrix} R_{BC_1} \\ R_{BC_2} \\ R_{S_1} \\ R_{F_1} \\ R_{U_1} \\ \vdots \\ R_{S_{N-1}} \\ R_{F_{N-1}} \\ R_{U_{N-1}} \\ R_{BC_3} \end{bmatrix}$$

$\delta\beta_u$ must be obtained from one more equation..

SOLUTION OF FALKNER-SKAN EQUATION BY FINITE DIFFERENCES CONT'D

β_u is a global variable which in general appears in each equation and thus disrupts the block-tridiagonal structure of the Jacobian matrix. This problem is sidestepped by writing the Newton system as:

$$\begin{bmatrix} A_1 & C_1 & & & & \\ B_2 & A_2 & C_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & B_i & A_i & C_i & \\ & & & \ddots & \ddots & \ddots \\ & & & C_{N-1} & A_{N-1} & C_{N-1} \\ & & & & B_N & A_N \end{bmatrix} \times \begin{bmatrix} \bar{\delta}_1 \\ \bar{\delta}_2 \\ \vdots \\ \bar{\delta}_i \\ \vdots \\ \bar{\delta}_{N-1} \\ \bar{\delta}_N \end{bmatrix} = - \begin{bmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \vdots \\ \bar{R}_i \\ \vdots \\ \bar{R}_{N-1} \\ \bar{R}_N \end{bmatrix} - \delta\beta_u \begin{bmatrix} \bar{S}_1 \\ \bar{S}_2 \\ \vdots \\ \bar{S}_i \\ \vdots \\ \bar{S}_{N-1} \\ \bar{S}_N \end{bmatrix}$$

A_i, B_i, C_i = 3×3 blocks

$$\bar{\delta}_i = \begin{bmatrix} \delta F_i \\ \delta U_i \\ \delta S_i \end{bmatrix}, \quad \bar{R}_i = \begin{bmatrix} R_{F_{i-1}} \\ R_{U_{i-1}} \\ R_{S_i} \end{bmatrix}, \quad \bar{S}_i = \begin{bmatrix} 0 \\ 0 \\ \partial R_{S_i}/\partial\beta_u \end{bmatrix}$$

This is now easily solved by a standard block-tridiagonal solution routine with two righthand sides to give:

$$\bar{\delta}_i = -\bar{r}_i - \delta\beta_u \, \bar{s}_i \qquad ; \quad 1 \le i \le N \qquad (*)$$

One additional equation is necessary to determine $\delta\beta_u$ and hence also $\bar{\delta}_i$. Two possibilities are:

either 1) $\beta_u - \beta_{u_{specified}} \equiv R_\beta = 0 \quad \Rightarrow \quad \delta\beta_u \left[\frac{\partial R_\beta}{\partial \beta_u}\right] = -R_\beta$

or 2) $H \int_0^{\eta_e} (1-U)U \, d\eta - \int_0^{\eta_e} (1-U)\, d\eta \equiv R_{\beta_u} = 0$; H = specified shape parameter

in discrete form: $$\sum_{i=1}^{N-1} \left\{ H\left[1 - \tfrac{1}{2}(U_{i+1}+U_i)\right]\tfrac{1}{2}(U_{i+1}+U_i) - \left[1 - \tfrac{1}{2}(U_{i+1}+U_i)\right] \right\} \Delta\eta_{i+1/2} \equiv R_\beta = 0$$

$$\Rightarrow \quad \sum_{i=1}^{N-1} \left(\delta U_{i+1} + \delta U_i\right)\left[\tfrac{1}{2}H - \tfrac{1}{2}H(U_{i+1}+U_i) + \tfrac{1}{2}\right] \Delta\eta_{i+1/2} = -R_\beta$$

Before this can be solved for $\delta\beta_u$, δU_{i+1} and δU_i must be expressed in terms of $\delta\beta_u$ only by using (*) above.

$$\bar{\delta}_i \equiv \begin{bmatrix} \delta F_i \\ \delta U_i \\ \delta S_i \end{bmatrix} = - \begin{bmatrix} r_{F_i} \\ r_{U_i} \\ r_{S_i} \end{bmatrix} - \beta_u \begin{bmatrix} s_{F_i} \\ s_{U_i} \\ s_{S_i} \end{bmatrix} \qquad \text{same as } (*)$$

Since r_{u_i} and s_{u_i} are known numbers at this point, δU_i and $\delta\beta_u$ are trivially related via (*)

Application of Newton-Raphson Method to Discrete Falkner-Skan Equations

Non-Linear System to be Solved. $3N+1$ unknowns: F_i, U_i, S_i, $(1 \le i \le N)$, β_u

$$\left.\begin{aligned}
&R_{F_i}\left(F_i \; U_i \; F_{i+1} \; U_{i+1}\right) \equiv F_{i+1} - F_i - \frac{\Delta\eta}{2}\left(U_{i+1} + U_i\right) = 0 \\
&R_{U_i}\left(U_i \; S_i \; U_{i+1} \; S_{i+1}\right) \equiv U_{i+1} - U_i - \frac{\Delta\eta}{2}\left(S_{i+1} + S_i\right) = 0 \\
&R_{S_i}\left(F_i \; U_i \; S_i \; F_{i+1} \; U_{i+1} \; S_{i+1} \; \beta_u\right) \\
&\qquad \equiv S_{i+1} - S_i + \frac{1+\beta_u}{2}\frac{\Delta\eta}{2}\left(F_{i+1}S_{i+1} + F_i S_i\right) + \beta_u \Delta\eta\left(1 - \tfrac{1}{2}\left(U_{i+1}^2 + U_i^2\right)\right) = 0
\end{aligned}\right\} \quad 1 \le i \le N-1$$

$$R_{BC_1}\left(F_1\right) \equiv F_1 = 0$$

$$R_{BC_2}\left(U_1\right) \equiv U_1 = 0$$

$$R_{BC_3}\left(U_N\right) \equiv U_N - 1 = 0$$

if β_u specified

if H specified

either $\quad R_\beta\left(\beta_u\right) \equiv \beta_u - \left(\beta_u\right)_{spec} = 0$

or $\quad R_\beta\left(U_1 \; U_2 \ldots U_i \ldots U_N\right) \equiv \sum_{i=1}^{N-1}\left(1 - \frac{U_{i+1}+U_i}{2}\right)\Delta\eta - \left(H\right)_{spec}\sum_{i=1}^{N-1}\left(1 - \frac{U_{i+1}+U_i}{2}\right)\left(\frac{U_{i+1}+U_i}{2}\right)\Delta\eta = 0$

Solution by Iteration: $F_i^{n+1} = F_i^n + \delta F_i$, $U_i^{n+1} = U_i^n + \delta U_i$, ... $\beta_u^{n+1} = \beta_u^n + \delta\beta_u$ etc.

The changes $\delta F_i \; \delta U_i \; \delta S_i \; \delta\beta_u$ are determined by the requirement that all the "residuals" $R_{F_i} \; R_{U_i} \ldots R_{BC_1} \ldots R_\beta$ at the next iteration be zero. For instance:

$$R_{F_i}^{n+1} \equiv R_{F_i}\left(F_{i+1}^{n+1} \; U_{i+1}^{n+1} \; F_i^{n+1} \; U_i^{n+1}\right) = R_{F_i}\left(F_{i+1}^n + \delta F_{i+1} \;\; U_{i+1}^n + \delta U_{i+1} \;\; F_i^n + \delta F_i \;\; U_i^n + \delta U_i\right)$$

$$\simeq R_{F_i}^n + \left(\frac{\partial R_{F_i}}{\partial F_{i+1}}\right)^n \delta F_{i+1} + \left(\frac{\partial R_{F_i}}{\partial U_{i+1}}\right)^n \delta U_{i+1} + \left(\frac{\partial R_{F_i}}{\partial F_i}\right)^n \delta F_i + \left(\frac{\partial R_{F_i}}{\partial U_i}\right)^n \delta U_i = 0$$

$$R_{S_i}^{n+1} \equiv R_{S_i}\left(F_{i+1}^{n+1} \; U_{i+1}^{n+1} \; S_{i+1}^{n+1} \; F_i^{n+1} \; U_i^{n+1} \; S_i^{n+1} \; \beta_u^{n+1}\right) = R_{S_i}\left(F_{i+1}^n + \delta F_{i+1} \;\; \ldots \;\; \beta_u^n + \delta\beta_u\right)$$

$$\simeq \underbrace{R_{S_i}^n}_{\text{Newton system righthand side}} + \underbrace{\left(\frac{\partial R_{S_i}}{\partial F_{i+1}}\right)^n}_{\text{Jacobian matrix coefficients}} \delta F_{i+1} + \left(\frac{\partial R_{S_i}}{\partial U_{i+1}}\right)^n \delta U_{i+1} + \ldots \left(\frac{\partial R_{S_i}}{\partial S_i}\right)^n \delta S_i + \left(\frac{\partial R_{S_i}}{\partial \beta_u}\right)^n \delta\beta_u = 0$$

Coefficient examples: $\quad \left(\frac{\partial R_{F_i}}{\partial U_{i+1}}\right)^n = -\frac{\Delta\eta}{2} \qquad \left(\frac{\partial R_{S_i}}{\partial S_i}\right)^n = -1 + \frac{1+\beta_u^n}{2}\frac{\Delta\eta}{2}F_i^n$

$$\left(\frac{\partial R_{S_i}}{\partial \beta_u}\right)^n = \frac{\Delta\eta}{4}\left(F_{i+1}^n S_{i+1}^n + F_i^n S_i^n\right) + \Delta\eta\left(1 - \tfrac{1}{2}\left({U_{i+1}^n}^2 + {U_i^n}^2\right)\right)$$

TREATMENT OF GLOBAL VARIABLE β_u

The standard Newton System will have the form:

$$
\begin{array}{c}
\delta F_1 \;\; \delta U_1 \;\; \delta S_1 \;\; \cdots \;\; \delta F_N \;\; \delta U_N \;\; \delta S_N \;\; \delta\beta_u \\
\begin{bmatrix}
\ddots & \ddots & & & & | \\
 & \ddots & \ddots & 0 & & | \\
 & & \ddots & \ddots & & | \\
 & 0 & & \ddots & \ddots & | \\
 & & & & \ddots & | \\
- & - & - & - & - &
\end{bmatrix}
\end{array}
\times
\begin{bmatrix} \delta F_1 \\ \delta U_1 \\ \delta S_1 \\ \vdots \\ \delta F_N \\ \delta U_N \\ \delta S_N \\ \delta\beta_u \end{bmatrix}
= -
\begin{bmatrix} R_{BC_1} \\ R_{BC_2} \\ R_{S_1} \\ R_{F_1} \\ \vdots \\ R_{U_{N-1}} \\ R_{BC_3} \\ R_\beta \end{bmatrix}
$$

Note that β_u in general appears in every equation, it has non-zero entries in its whole column. Likewise, the the residual R_β is in general a function of all the variables, and hence it has non-zero entries in its whole row. This spoils the diagonal structure of the coefficient matrix, making its solution awkward.

This problem is eliminated by solving the system in two steps.

Step 1: The last equation (bottom matrix row) is put aside, and the $\delta\beta_u$ column is placed on the righthand side. This is now easily solved.

$$
\begin{array}{c}
\delta F_1 \;\; \delta U_1 \;\; \delta S_1 \;\; \cdots \;\; \delta F_N \;\; \delta U_N \;\; \delta S_N \\
\begin{bmatrix}
\ddots & \ddots & & & \\
 & \ddots & \ddots & 0 & \\
 & & \ddots & \ddots & \\
 & 0 & & \ddots & \ddots \\
 & & & & \ddots
\end{bmatrix}
\end{array}
\times
\begin{bmatrix} \delta F_1 \\ \delta U_1 \\ \delta S_1 \\ \vdots \\ \delta F_N \\ \delta U_N \\ \delta S_N \end{bmatrix}
= -
\begin{bmatrix} R_{BC_1} \\ R_{BC_2} \\ R_{S_1} \\ R_{F_1} \\ \vdots \\ R_{U_{N-1}} \\ R_{BC_3} \end{bmatrix}
- \delta\beta_u
\begin{array}{c} \delta\beta_u \\ \begin{bmatrix} | \\ | \\ | \end{bmatrix} \end{array}
\xrightarrow[\text{Solver}]{\text{Block Matrix}}
\begin{bmatrix} \delta F_1 \\ \delta U_1 \\ \delta S_1 \\ \vdots \\ \delta F_N \\ \delta U_N \\ \delta S_N \end{bmatrix}
= -
\begin{bmatrix} r_{F_1} \\ r_{U_1} \\ r_{S_1} \\ \vdots \\ r_{F_N} \\ r_{U_N} \\ r_{S_N} \end{bmatrix}
- \delta\beta_u
\begin{bmatrix} s_{F_1} \\ s_{U_1} \\ s_{S_1} \\ \vdots \\ s_{F_N} \\ s_{U_N} \\ s_{S_N} \end{bmatrix}
\qquad (*)
$$

Step 2:
It is still necessary to find $\delta\beta_u$ so that δF_i δU_i δS_i can be completely determined. This is done by using the last equation which was put aside:

$$
\begin{bmatrix} \dfrac{\partial R_\beta}{\partial F_1} & \dfrac{\partial R_\beta}{\partial U_1} & \cdots & \dfrac{\partial R_\beta}{\partial S_N} \end{bmatrix}
\times
\begin{bmatrix} \delta F_1 \\ \delta U_1 \\ \vdots \\ \delta S_N \end{bmatrix}
+ \begin{bmatrix} \dfrac{\partial R_\beta}{\partial \beta_u} \end{bmatrix} \delta\beta_u = -R_\beta
$$

By substituting (*):

$$
-\begin{bmatrix} \dfrac{\partial R_\beta}{\partial F_1} & \dfrac{\partial R_\beta}{\partial U_1} & \cdots & \dfrac{\partial R_\beta}{\partial S_N} \end{bmatrix}
\times
\begin{bmatrix} s_{F_1} \\ s_{U_1} \\ \vdots \\ s_{S_N} \end{bmatrix} \delta\beta_u
+ \begin{bmatrix} \dfrac{\partial R_\beta}{\partial \beta_u} \end{bmatrix} \delta\beta_u
= -R_\beta +
\begin{bmatrix} \dfrac{\partial R_\beta}{\partial F_1} & \dfrac{\partial R_\beta}{\partial U_1} & \cdots & \dfrac{\partial R_\beta}{\partial S_N} \end{bmatrix}
\times
\begin{bmatrix} r_{F_1} \\ r_{U_1} \\ \vdots \\ r_{S_N} \end{bmatrix}
$$

This is a scalar system for $\delta\beta_u$. Knowing $\delta\beta_u$, δF_i δU_i δS_i are obtained from (*).