Lecture 8

Thin Shear Layer Approximation

3.1) A) TSL Equations: Summary, Edge Conditions, Coordinates, Streamfunction

B) Shear Layer Categories and Boundary conditions

C) Axisymmetric form.

Reading:

A)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{unsteady}} + \frac{u\partial u}{\partial x} + \frac{v\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial p}{\partial y} = 0$$

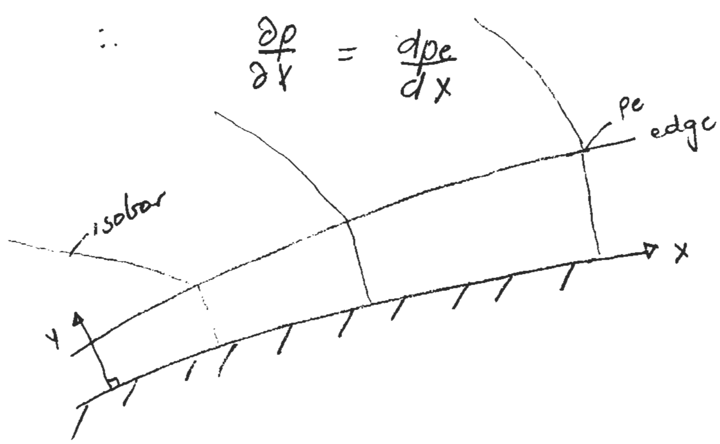
Pressure at the edge of the TSL:

$$\frac{\partial p}{\partial y} \approx 0$$

$$\Rightarrow p(x,y) = p_e(x)$$

where e denotes edge of shear layer

$$\therefore \frac{\partial p}{\partial x} = \frac{dp_e}{dx}$$



Recall Bernoulli's Equation for steady incompressible inviscid flow can be applied along a streamline

$$p_e(x) + \tfrac{1}{2}\rho_e (u_e^2 + v_e^2) = p_0$$

$$\Rightarrow \frac{dp_e}{dx} = -\rho_e u_e \frac{du_e}{dx} - \rho_e v_e \frac{dv_e}{dx}$$

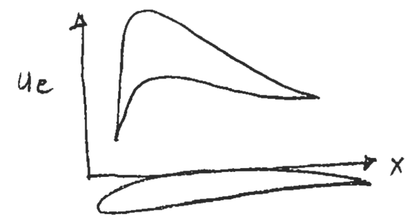
since $v_e << u_e$

$$\frac{dp_e(x)}{dx} = -\rho_e u_e \frac{du_e}{dx}$$

compressible (drop for incomp) (can also be obtained by considering x-mom for outer flow)

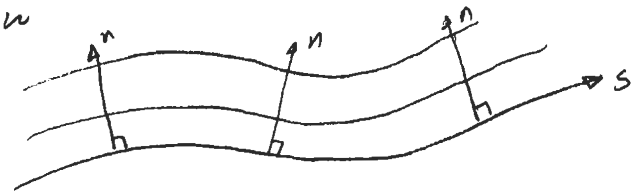
(*) Now, given edge conditions, we can solve TSL equations to obtain b. layer behavior

Example



reasonable approx. in (unseparated flow)

TSL coordinates are not Cartesian. They are typically streamwise and normal coordinates (s & n), oriented so that $v << u$



(*) Note Incomp vs compressible form of x-mom

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e \frac{du_e}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y} \quad \text{where} \quad \tau = \mu \frac{\partial u}{\partial y}$$

" "

$$\rho u\frac{\partial u}{\partial y} + \rho v\frac{\partial u}{\partial y} = \rho_e u_e \frac{du_e}{dx} + \frac{\partial \tau}{\partial y}$$

TSL in streamfunction form (steady flows)

A streamfunction is a scalar function $\psi(x,y)$ s.t

$$u = \frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x}$$

Substituting in x-momentum eqn.

$$\frac{\partial\psi}{\partial y}\cdot\frac{\partial u}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial u}{\partial y} = u_e\frac{dU_e}{dx} + \nu\frac{\partial^2}{\partial y^2}\left(\frac{\partial\psi}{\partial y}\right)$$

$$\psi_y\cdot\psi_{xy} - \psi_x\,\psi_{yy} = u_e\frac{du_e}{dx} + \nu\,\psi_{yyy}$$

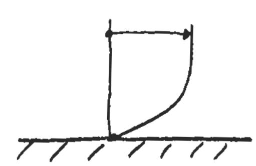
Note 1. continuity is no longer required
2. compressible streamfunction $\rho u = \psi_y$ $\rho v = -\psi_x$

B> Shear Layer Categories and B.Cs.

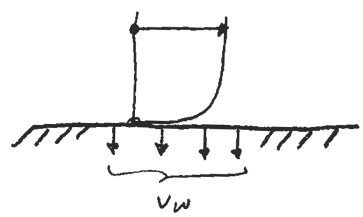
TSL equations apply to very wide variety of flows.
Different flows distinguished by boundary conditions

3rd-order system requiring 3 BCs per x location

1> Wall B.L: @ $y=0$: $u=0$, $v=0$
$y=y_e$ $u=u_e$



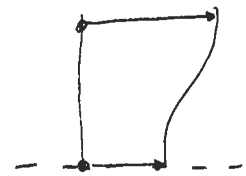
2> Porous wall: @ $y=0$: $u=0$ $v=v_w$
$y=y_e$ $u=u_e$



3) Wake or Jet (Symmetric) :

@ $y=0$: $\frac{\partial u}{\partial y}=0$, $v=0$

@ $y=y_e$: $u=u_e$



Wake

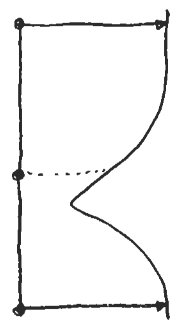
$V_e=0$ for jet

4) Wake (General) :

@ $y=y_e^+$ $\quad u=u_e^+$

@ $y=y_e^-$ $\quad u=u_e^-$

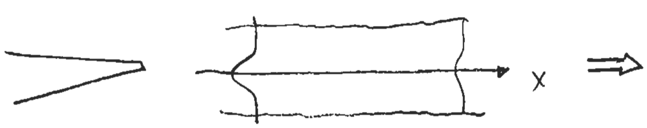
@ $y=y_i$ $\quad v=v_i$

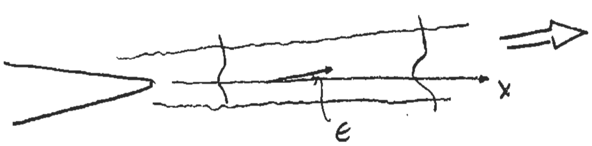


u_i is some arbitrary interior point, and $v_i=v(y_i)$ is also arbitrary, as long as $v_i \ll u_e$. Changing v_i merely repositions shear layer in x, y coordinate system

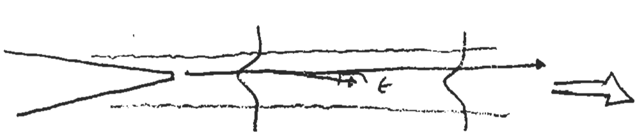
Example

$y=0$, $v=0$





$y=0$ $\quad v=\epsilon u$



$y=0$ $\quad v=-\epsilon u$

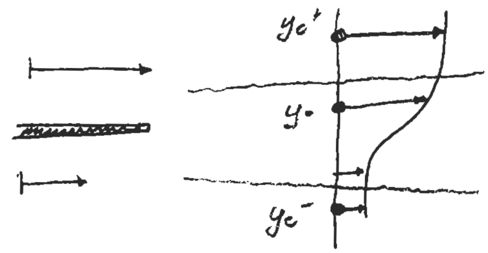
(x,y) coordinate is OK as long a x is closely aligned with TSL, so that $\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x}$ assumption is valid.

5) Mixing Layer

@ $y = y_e^+$: $u = u_e^+$

$y = y_0$: $v = v_0$

$y = y_e^-$: $u = u_e^-$



c) Axisymmetric TSL's

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right)$$

$$\frac{\partial p}{\partial y} = 0$$

$$\nabla \cdot \vec{u} = 0$$

$$p_e(x, z) \quad \text{and} \quad \vec{u}_e = u_e\hat{i} + w_e\hat{k}$$

z - spanwise direction on a swept wing (away from root)

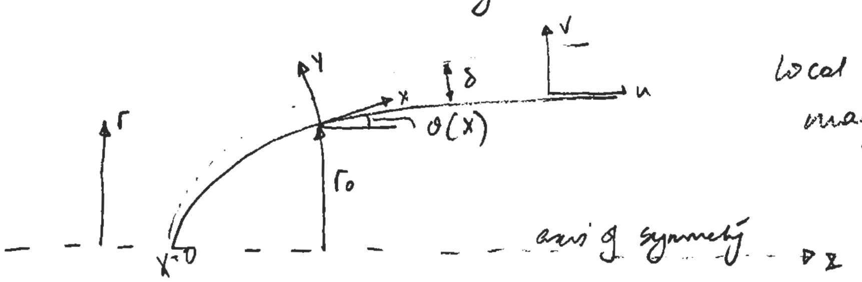
Special case of slender shear layer - axisymmetric body - gradients around the circumference are zero (flow in a duct, wing-body junction

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho r}\frac{\partial}{\partial y}\left(r\left(\mu\frac{\partial u}{\partial y}\right)\right)$$

$$\frac{\partial}{\partial x}\left(r^k u\right) + \frac{\partial}{\partial y}(r v) = 0$$

Note $\theta(x)$ or δ/r_0 need not be small

local streamline curvature may result in $\frac{\partial p}{\partial y} \neq 0$



$$r = r_0 + y\cos\theta \qquad \tan\theta = \frac{dr_0}{dz}$$

Define $\quad t \equiv \dfrac{y\cos\theta}{r_0} \quad \Rightarrow \quad \dfrac{r}{r_0} = 1 + t$

↑ transverse curvature term

If $\quad t \ll 1 \qquad \delta/r_0 \ll 1$

Eqn simplify to $\quad \dfrac{\partial}{\partial x}(r_0 u) + \dfrac{\partial}{\partial y}(r_0 v) = 0$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \underbrace{-\frac{1}{\rho}\frac{dp}{dx}}_{+u_e\frac{du_e}{dx}} + \underbrace{\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)}_{\nu\frac{\partial^2 u}{\partial y^2}}$$

Does not apply to axisymmetric jet $r \to 0$ – axis of symmetry. In that case,

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial r}(rv) = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \underbrace{-\frac{1}{\rho}\frac{dp}{dx}}_{u_e\frac{du_e}{dx}} + \underbrace{\frac{1}{\rho r}\frac{\partial}{\partial r}\left(r\mu\frac{\partial u}{\partial r}\right)}_{\nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)}$$

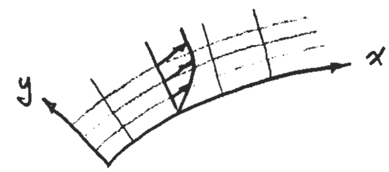
Ax. Jet $\quad$ B.C $\quad r=0,\ v=0,\ \dfrac{\partial u}{\partial r}=0, \quad r=r_e,\ u=0$

Thin Shear Layer Equations and Boundary Conditions

TSL Approximations: $\dfrac{\partial^2 u}{\partial y^2} \gg \dfrac{\partial^2 u}{\partial x^2}$, transvere diffusion $\gg$ streamwise diffusion

$\dfrac{\partial p}{\partial x} \simeq$ constant in y, so $p(x,y) = p_e(x)$, $\dfrac{\partial p}{\partial x} = \dfrac{dp_e}{dx} = -\rho u_e \dfrac{du_e}{dx}$

TSL Equations: Continuity: $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

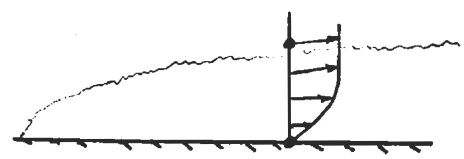


x-Momentum: $u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = u_e \dfrac{du_e}{dx} + \nu \dfrac{\partial^2 u}{\partial y^2}$

Boundary Conditions: 3rd-order system, needs 3 BCs per x location

Wall Boundary Layer

1) $y = y_e$: $u = u_e$ or $v = v_e$
2) $y = 0$: $u = 0$
3) $y = 0$: $v = 0$

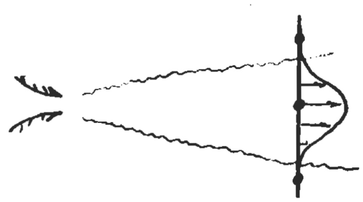


Wake

1) $y = y_{e+}$: $u = u_e$
2) $y = y_o$: $v = V_0$
3) $y = y_{e-}$: $u = u_e$

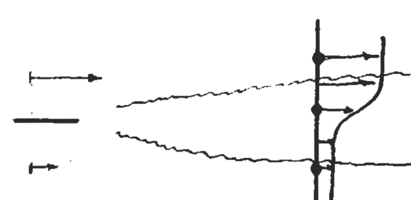
Jet

1) $y = y_{e+}$: $u = 0$
2) $y = y_o$: $v = V_0$
3) $y = y_{e-}$: $u = 0$



Mixing Layer

1) $y = y_{e+}$: $u = u_{e+}$
2) $y = y_o$: $v = V_0$
3) $y = y_{e-}$: $u = u_{e-}$



Porous Wall B.L.

1) $y = y_e$: $u = u_e$
2) $y = 0$: $u = 0$
3) $y = 0$: $v = v_w$